Reconsidering the Oil-Refinery Data

- Clearly, level responds to reflux flow.
- Also clearly not a concurrent linear model.
- In fact, no linear model will capture this relationship.
- But, there is clearly something with fairly simple structure going on.

A Thought on the Concurrent Linear Model

- Still not linear; include the current value in the tray.
- This gives us the following concurrent linear model:

\[ Dx(t) = \beta x(t) + \alpha u(t) \]

One repetition – must borrow strength across time and make \( \alpha, \beta \) constant.
- This is now a differential equation.

Try relating rate of change to input.

Differential Equations

Simple linear differential equation model

\[ Dx(t) = \beta x(t) + \alpha u(t) \]

- Constant coefficient concurrent linear model
- Dependent variable is the derivative of system output \( x(t) \).
- two independent variables
  - output function \( x(t) \) itself
  - input function \( u(t) \).
- Model for change.

Even More Simply

- Lets assume there is no input \( u(t) \).
- The equation is

\[ Dx(t) = \beta x(t) \]

and is called homogeneous.

This equation describes how change (\( Dx(t) \)) depends only on the current value of \( x(t) \).

It describes the internal dynamics of a system, or how the system would behave if left alone.

When we add in \( u(t) \), we have an inhomogeneous system that describes the impact of external influences.
Why Dynamics?

Three reasons for using models for dynamics

1. Captures seemingly-complex relationships to be broken down into simple models.

2. For repeated curves it allows
   - Use of concurrent linear models in functional data analysis
   - Partition of variation between variation in the process producing the curve and observation noise.
   - Frequently, it is easiest to think about the processes behind the curve in terms of rates of change.

Thinking About Models for Rates

Imagine a bucket with a hole in the bottom.

- left to itself, the water will flow out the hole and the level will drop
- adding water will increase the level in the bucket
- we want to describe the rate at which this happens

Water in a leaky bucket.

So far

\[ Dx(t) = \beta x(t) \]

If we add liquid at rate \( u(t) \), we simply increase \( Dx(t) \) by this amount:

\[ Dx(t) = \beta x(t) + u(t) \]

Since the bucket is cylindrical, volume is proportional to height and we can use \( x(t) \) for height instead.

Water in a leaky bucket.

To make things simple, assume the bucket has straight sides. Let \( x(t) \) be the current volume of liquid in the bucket.

- Firstly, we need a rate for outflow without input.
  - The rate at which water leaves the bucket is proportional to how much pressure it is under.
    \[ Dx(t) = -Cp(t) \]
  - The pressure will be proportional to the weight of liquid. This in turn is proportional to volume.
    \[ Dx(t) = \beta x(t) \]
Solutions to Homogeneous First-Order Differential Equations

Let's turn off the tap:

$$Dx(t) = \beta x(t)$$

- All $x(t)$ satisfying this are of the form
  $$x_0(t) = Ce^{\beta t}$$
  (remember that $\beta < 0$ in our bucket)
- If we know the value of $x_0(t)$ at $t = 0$ then $C = x_0(0)$.
- Left to itself the system will
  - decay to zero if $\beta < 0$
  - grow exponentially quickly if $\beta > 0$
  - remain unchanged if $\beta = 0$.

Adding Inhomogeneous Terms

When the tap is turned back on:

$$Dx(t) = \beta x(t) + \alpha u(t)$$

solutions are of the form

$$x(t) = Ce^{\beta t} + \alpha \int_0^t e^{(t-s)\beta} u(s) \, ds$$

As for homogeneous equation, the constant $C$ is $x(0)$.

This formula is not particularly enlightening; we would like to investigate how $x(t)$ behaves.

Characterizing Solutions to Step-Function Inputs

In engineering, it is common to study the reaction of $x(t)$ when $u(t)$ is abruptly stepped up or down.

Let's simplify to $x(0) = C = 1$ and let $u(t) = 0$ for $0 \leq t \leq 1$ and $u(t) = 1$ for $t > 1$.

Then $x(t)$ is given by

$$x(t) = \begin{cases} e^{\beta t} & 0 \leq t \leq 1 \\ e^{\beta t} + \left(\frac{\alpha}{\beta}\right) \left[1 - e^{\beta(t-1)}\right] & t > 1 \end{cases}$$

- On $t \leq 1$, $x(t)$ decays towards zero. It gets to 0.02 in about $-4/\beta$ time units.
- When $u(t)$ is stepped up, the solution grows at an exponentially decreasing rate, eventually reaching $\alpha/\beta$, often called the gain of the system.
- 98% Gain is achieved in about $-4/\beta$ time units.
- $\beta$ controls responsivity of the system
- $\alpha$ controls the gain.
Response to Step Functions
Set $\alpha = 4\beta$

![Graph showing response to step functions](image)

Fit to Oil Refinery Data
Set $\alpha = -0.19$, $\beta = -0.02$

![Graph showing fit to oil refinery data](image)

Nonconstant Coefficients

What if the size of the hole changes over time?
Can write the non-constant coefficient homogeneous equation as

$$Dx(t) = \beta(t)x(t)$$

with solution of the form

$$x_0(t) = C \exp \left[ \int_0^t \beta(s) ds \right]$$

Think of $\beta(t)$ as instantaneous reponsivity.

Nonconstant Coefficients

For the inhomogeneous system

$$Dx(t) = \beta(t)x(t) + \alpha(t)u(t)$$

solution is

$$x(t) = Ce_0^t \beta(s) ds + e_0^t \beta(s) ds \int_0^t \alpha(s) u(s) e_0^\nu \beta(v) dv ds$$

$\alpha$ is instantaneous gain.

When $\alpha(t)$ and $\beta(t)$ vary more slowly than $x(t)$ it is easiest to think of $x(t)$ behaving (locally) as though they were constant.
Beyond First-order Systems

There is no reason that only first derivatives should be modeled. Most famous differential equation comes from Newton’s first law:

\[ D^2 x(t) = u(t) / m \]

force = mass * acceleration

More generally, we will consider models of the form:

\[ D^2 x(t) = \beta_1(t)Dx(t) + \beta_0(t)x(t) + \alpha(t)u(t) \]

We will see that adding more derivatives allows responses to occur on multiple time-scales.

Multi-Dimensional Systems

There is also no reason to restrict attention to only one dimension. For two-dimensional systems we can model

\[ Dx(t) = \beta_{11}(t)x(t) + \beta_{12}(t)y(t) + \alpha_1(t)u_1(t) \]
\[ Dy(t) = \beta_{21}(t)x(t) + \beta_{22}(t)y(t) + \alpha_2(t)u_2(t) \]

or we can use higher-order derivatives as well. This makes interpretation harder.

Second Order Systems: Physical Springs

We can imagine a weight at the end of a spring. For simple mechanics

\[ D^2 x(t) = f(t) / m \]

here the force, \( f(t) \), is a sum of components

1. \(-\beta_0(t)x(t)\): the force pulling the spring back to rest position.
2. \(-\beta_1(t)Dx(t)\): forces due to friction in the system
3. \(\alpha(t)u(t)\): external forces driving the system

Springs make good initial models for physiological processes, too.

Solutions to One-Dimensional Systems

Generally, if we have

\[ D^m x(t) = \beta_{m-1} D^{m-1} x(t) + \ldots + \beta_0 x(t) \]

the solution is of the form

\[ x(t) = C_1 e^{\lambda_1 t} + \ldots + C_m e^{\lambda_m t} \]

where the \( \lambda \) are solutions to

\[ \lambda^m = \beta_{m-1} \lambda^{m-1} \ldots + \beta_0 \]

This says that there are \( m \) different timescales at which the system responds.
Solutions to One-Dimensional Second-Order Systems

We need solutions to

\[ \lambda^2 = \beta_1 \lambda + \beta_0 \]

What if these do not exist? Eg \( \lambda^2 = -1 \).

Solutions are of the form

\[ x_0(t) = C_1 e^{\lambda_1 t} \cos(\lambda_2 t) + C_2 e^{\lambda_1 t} \sin(\lambda_2 t) \]

these produce oscillations that either increase exponentially in magnitude, or decrease exponentially.

Responses to input can be more difficult to determine.

Example

\[ D^2 x(t) = -0.4Dx(t) - 2.08x(t) + u(t) \]

Here \( \lambda_1 = 0.2, \lambda_2 = 1 \).

The Discriminant Function

Constant co-efficient solutions are of the form:

\[ \lambda = \frac{\beta_1}{2} \pm \sqrt{d} \]

with the discriminant being

\[ d = \frac{\beta_1^2}{2} + \beta_0 \]

- If \( d < 0 \) the system oscillates with growing or shrinking cycles according to the sign of \( \beta_1 \).
- If \( d > 0 \) the system is under-damped
  - If \( \beta_1 > 0 \) or \( \beta_0 > 0 \) the system exhibits exponential growth.
  - If \( \beta_1 < 0 \) and \( \beta_0 < 0 \) the system decays exponentially.

Graphically

This means we can partition \((\beta_0, \beta_1)\) space into regions of different qualitative dynamics.

This is known as a bifurcation diagram.

Time-varying dynamics. Like constant-coefficient dynamics at each time, if \( \beta_1(t), \beta_0(t) \) evolve more slowly than \( x(t) \).
Higher-Dimensional Systems

We write a first-order \( k \)-dimensional system in general terms as

\[
Dx(t) = A(t)x(t) + B(t)u(t)
\]

where \( A(t) \) and \( B(t) \) are \( k \)-by-\( k \) and \( k \)-by-\( p \) matrices.

Stability for homogeneous part depends on eigenvalues of \( A(t) \).

- Any eigenvalues with imaginary parts imply oscillation.
- If real part of corresponding eigenvalue is positive, oscillations increase, otherwise they die out.
- If real part of any eigenvalue is positive, the system explodes exponentially. Otherwise it shrinks.

Example

For the system

\[
D^2x(t) = -0.4Dx(t) - 2.08x(t) + u(t)
\]

\( Dx(t) \) is like a second dimension. We can plot these together:

Here, \( \lambda = 0.2 \pm \sqrt{-2.07} \).
This is called a phase-plane plot.
Highlights stability properties of the system.
Can also get expanding spirals, stable ellipses, and lines that converge or diverge.

Higher-Order Higher-Dimensions

What about second order systems?

\[
D^2x(t) = A_1(t)Dx(t) + A_0(t)x(t) + B(t)u(t)
\]

we let \( y(t) = Dx(t) \) then

\[
\begin{pmatrix}
Dx(t) \\
Dy(t)
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
A_1(t) & A_0(t)
\end{pmatrix} \begin{pmatrix}
y(t) \\
x(t)
\end{pmatrix} + \begin{pmatrix}
0 \\
B(t)
\end{pmatrix} u(t)
\]

Same eigenvalue rules apply to this system.

If \( u(t) \) varies slowly relative to \( x \), solutions just add \( B(t)u \) to the homogeneous solution.

Nonlinear Dynamics

Far more generally, we can write

\[
Dx(t) = f(x(t), u, \theta)
\]

where \( f \) is some, possibly non-linear function.

Such models make up a large part of applied mathematics.

Models are not solvable analytically, but can produce complex behavior:

- cycles
- points of attraction
- unstable orbits
- switches between them (bifurcations)
Measles Models

Susceptible (S), Exposed (E), Infected (I), Recovered (R) Individuals

\[
\begin{align*}
DS(t) &= \nu - \beta(t)S(t)I(t) - \mu S(t) \\
DE(t) &= \beta(t)S(t)I(t) - \sigma E(t) - \mu E(t) \\
DI(t) &= \sigma E(t) - \gamma I(t) - \mu I(t) \\
DR(t) &= \gamma I(t) - \mu R(t)
\end{align*}
\]

- \( \nu \) = births
- \( \beta(t) \) = transmission rate
- \( \mu \) = death rate
- \( \sigma \) = rate of transition \( E \rightarrow I \)
- \( \gamma \) = rate of transition \( I \rightarrow R \)

Summary

- Dynamic models are useful for
  - developing physical models for functional data
  - allowing more models to be fit by linear processes
  - improving smoothing of data (we’ll see more later)
- solutions for linear dynamics in terms of exponential and trig functions
- stability of dynamical systems in the same terms
- Nonlinear dynamics has potential for vastly more powerful modeling; almost no work in statistics.