

RCB design with replication

$$R_{(ij)k}$$

$$Y_{ijk} = \mu_0 + B_i + \gamma_j + B\gamma_{ij} + E_{ijk}$$

$$\gamma: A_3 = \bar{I}_b \otimes C_t \otimes \bar{I}_r$$

$$B\gamma: A_4 = \bar{I}_b \otimes C_t \otimes \bar{I}_r$$

$$\Sigma = \sigma_E^2 I_5 + (\sigma_E^2 + r\sigma_{B\gamma}^2)(A_3 + A_4) \\ + (\sigma_E^2 + r\sigma_{B\gamma}^2 + b\sigma_B^2)(A_1 + A_2)$$

$$Y'A_3Y \sim (\sigma_E^2 + r\sigma_{B\gamma}^2) \chi_{t-1}^2(\delta_3)$$

$$\delta_3 = \frac{b r \sum (y_i - \bar{y})^2}{\sigma_E^2 + r\sigma_{B\gamma}^2}$$

$$Y'A_4Y \sim (\sigma_E^2 + r\sigma_{B\gamma}^2) \chi_{(b-1)(t-1)}^2(0)$$

t and F-distributions

- If $y \sim N(\alpha, 1)$ indep. of $u \sim \chi_n^2(0)$

$$\text{then } T = \frac{y}{\sqrt{u/n}}$$

has a non-central t-distribution
with n degrees of freedom and non-centrality parameter α

$$\text{write } T \sim t_n(\alpha)$$

- If $U_1 \sim \chi^2_{n_1}(\delta)$ indep. of $U_2 \sim \chi^2_{n_2}(0)$

Then $F = \frac{U_1/n_1}{U_2/n_2}$

has a F-distribution with n_1 and n_2 degrees of freedom and noncentrality δ

Write $F \sim F_{n_1, n_2}(\delta)$

- Note that if $\gamma \sim N(\alpha, 1)$ then

$$\gamma^2 \sim \chi^2_{\alpha^2}(\alpha^2)$$

- $t_n(\alpha^2) \sim F_{1, n}(\alpha^2)$

- RCB design

$$\frac{\gamma' A_3 \gamma / (t-1)}{\gamma' A_4 \gamma / (b-1)(t-1)} \sim F_{t-1, (b-1)(t-1)}(\delta_3)$$

$$\delta_3 = \frac{b \sum_j \gamma_j^2}{\sigma_B^2 + \sigma_E^2 / r}$$

Expected Mean Squares

Define EMS for the i th term as

$$E \left\{ \frac{\gamma' A_i \gamma}{\text{tr}(A_i)} \right\} = E \left\{ \frac{\gamma' A_i \gamma}{\text{rank}(A_i)} \right\} \text{ if } A_i = A_i^2$$

Under the conditions of Bhat's Lemma

$$E(Y' A_i Y) = c_i E(\chi^2_{n_i}(\delta_i))$$

$$= c_i (n_i + \delta_i)$$

$$EMS_i = c_i (1 + \frac{1}{n_i} \delta_i) = c_i + \frac{\mu' A_i \mu}{n_i}$$

e.g. RCB design

$$EMS_3 = \sigma_E^2 + r \sigma_{B\tau}^2 + \frac{b \sigma \sum \tau_j^2}{t-1}$$

$$EMS_4 = \sigma_E^2 + r \sigma_{B\tau}^2$$

Finite Model

- RCB design
- same t treatments in every block
- $E(B\tau_{ij}) = 0$ average block/treatment

interaction is zero for all i

$$E(B\tau_{ij}) = \frac{1}{t} \sum_{j=1}^t B\tau_{ij} = 0 \rightarrow \sum_{j=1}^t B\tau_{ij} = 0$$

- Assume $(B\tau_{i1}, \dots, B\tau_{it})'$ $i=1, \dots, b$ are independent but

$$(B\tau_{i1}, \dots, B\tau_{it})' \sim N_t(0, \sigma_{B\tau}^2 C_t)$$

$$\text{So } 1_t'(B\tau_{i1}, \dots, B\tau_{it})' = \sum_j B\tau_{ij}$$

$$\text{var}(\sum_j B\tau_{ij}) = \sigma_{B\tau}^2 1_t' C_t 1_t = 0$$

This implies

$$\begin{aligned}\Sigma &= \sigma_B^2 (I_b \otimes J_t \otimes J_r) \\ &\quad + \sigma_{BT}^2 (I_b \otimes C_t \otimes J_r) \\ &\quad + \sigma_E^2 (I_b \otimes I_t \otimes I_r)\end{aligned}$$

$$\text{Recall } C_t = I_t - \bar{J}_t$$

$$\begin{aligned}S_0 \Sigma &= (\sigma_B^2 - \frac{1}{t} \sigma_{BT}^2) (I_b \otimes I_t \otimes J_r) \\ &\quad + \sigma_{BT}^2 (I_b \otimes I_t \otimes J) \\ &\quad + \sigma_E^2 (I_b \otimes I_t \otimes I)\end{aligned}$$

If σ_{B*}^2 , σ_{BT*}^2 and σ_E^2 denote the variance components in the infinite model

then

$$\sigma_{B*}^2 = \sigma_B^2 - \frac{1}{t} \sigma_{BT}^2, \quad \sigma_{BT*}^2 = \sigma_{BT}^2, \quad \sigma_E^2 = \sigma_E^2$$

Single Factor expt. with replicates

$$\begin{aligned}Y_{ij} &= \mu_0 + \mu_i + \epsilon_{ij} (+ R_{ij}) \\ &= \mu_i + \epsilon_{ij} \quad \begin{matrix} i=1, \dots, t \\ j=1, \dots, r \end{matrix}\end{aligned}$$

μ_i = mean response to treatment i

$$\epsilon_{ij} \stackrel{iid}{\sim} N(0, \sigma_E^2)$$

• estimate μ_i by $\hat{\mu}_i = \bar{Y}_{i0}$

- average expected squared error is

$$E \frac{1}{t} \sum_{i=1}^t (\hat{\mu}_i - \mu_i)^2 = \frac{\sigma_E^2}{r}$$

- Consider $\tilde{\mu}_i = \bar{Y}_{i\cdot} - \rho (\bar{Y}_{\cdot i} - \bar{Y}_{..})$

for some $0 \leq \rho \leq 1$

Then

$$\begin{aligned} E \frac{1}{t} \sum_{i=1}^t (\tilde{\mu}_i - \mu_i)^2 &= \frac{\sigma_E^2}{r} \left[1 - 2 \rho \frac{t-1}{t} + \rho^2 \frac{\frac{1}{t} \sum (\mu_i - \bar{\mu})^2}{\sigma_E^2 / r} \right] \\ &\quad \bar{\mu} = \frac{1}{t} \sum \mu_i \end{aligned}$$

- Suppose $\rho = \frac{\sigma_E^2 / r}{\sigma_E^2 / r + \frac{1}{t} \sum (\mu_i - \bar{\mu})^2}$ then

$$E \frac{1}{t} \sum_{i=1}^t (\tilde{\mu}_i - \mu_i)^2 = \frac{\sigma_E^2}{r} \left[1 - \frac{t-2}{t} \rho - \frac{1}{t} \rho^2 \right] < \frac{\sigma_E^2}{r}$$

$$B = \sum_{i \in J} \sum_j (\bar{Y}_{i\cdot} - \bar{Y}_{..}) = Y' C_t \otimes \bar{J}_r Y$$

$$E(B/(t-1)) = \sigma_E^2 + \frac{r \sum (\mu_i - \bar{\mu})^2}{t-1}$$

$$W = \sum_{i,j} (Y_{ij} - \bar{Y}_{i\cdot})^2 = Y' (I_t \otimes C_r) Y$$

$$E(W/t(r-1)) = \sigma_E^2$$

$$\text{Use } \hat{\rho} = \frac{W/t(r-1)}{B/(t-1)}$$

Random single factor model

$$Y_{ij} = \mu_0 + T_i + E_{ij}$$

$$T_i \stackrel{iid}{\sim} N(0, \sigma_T^2), \quad E_{ij} \stackrel{iid}{\sim} N(0, \sigma_E^2)$$

$$\bar{Y}_{i..} = \mu_0 + T_i + \bar{E}_{i..}$$

$$\stackrel{iid}{\sim} N(\mu_0, \sigma_T^2 + \sigma_E^2/r)$$

$$\begin{pmatrix} \bar{Y}_{i..} \\ T_i \end{pmatrix} \stackrel{iid}{\sim} N_2 \left[\begin{pmatrix} \mu_0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_T^2 + \sigma_E^2/r & \sigma_T^2 \\ \sigma_T^2 & \sigma_T^2 \end{pmatrix} \right]$$

$$T_i | \bar{Y}_{i..} \sim N \left[0 + \frac{\sigma_T^2}{\sigma_T^2 + \sigma_E^2/r} (\bar{Y}_{i..} - \mu_0), \sigma_T^2 \left(1 - \frac{\sigma_T^2}{\sigma_T^2 + \sigma_E^2/r} \right) \right]$$

Best Linear Unbiased Predictor (BLUP)
of $\mu_0 + T_i$ is

$$\mu_i^* = \bar{Y}_{i..} + \frac{\sigma_T^2}{\sigma_T^2 + \sigma_E^2/r} (\bar{Y}_{i..} - \bar{Y}_{..})$$

$$E[B/(t-1)] = \sigma_E^2 + r \sigma_T^2$$

$$E[W/t(r-1)] = \sigma_E^2$$

- Estimate $\frac{\sigma_T^2}{\sigma_T^2 + \sigma_E^2/r}$ by $\frac{W/t(r-1)}{B/(t-1)}$