

Complete Balanced Factorial Designs

RCB Design

$$Y_{ijk} = \mu_0 + B_i + T_j + BT_{ij} + R(BT)_{(ij)k} + E_{ijk}$$

Block and treatment are crossed

- multiple obs. for each B/T combination

$$i=1, \dots, b, \quad j=1, \dots, t, \quad k=1, \dots, r$$

Block is a random factor - a sample from a population

Treatment is a fixed factor - same treatments in repetitions of the experiment

$$B_i \sim \text{iid } N(0, \sigma_B^2)$$

$$BT_{ij} \sim \text{iid } N(0, \sigma_{BT}^2)$$

$$E_{ijk} \sim \text{iid } N(0, \sigma_E^2)$$

Sum of squares: $\mathbf{Y}' A_\ell \mathbf{Y} \quad \ell = 1, \dots, 5 = \# \text{ terms}$

$A_1 \quad \mu_0$	$\bar{J}_b \otimes \bar{J}_t \otimes \bar{J}_r$	$\sum_{\ell=1}^5 A_\ell = I$
$A_2 \quad B$	$C_b \otimes \bar{J}_t \otimes \bar{J}_r$	
$A_3 \quad T$	$\bar{J}_b \otimes C_t \otimes \bar{J}_r$	$\sum_{\ell=1}^5 \text{rank}(A_\ell) = btr$
$A_4 \quad BT$	$C_b \otimes C_t \otimes \bar{J}_r$	

$$A_5 \leftarrow I_b \otimes I_t \otimes C_r$$

- Multivariate form

$$\begin{aligned} \tilde{Y} &= (I_b \otimes I_t \otimes I_r) \mu_0 \\ &+ (I_b \otimes I_t \otimes I_r) \tilde{B} \\ &+ (I_b \otimes I_t \otimes I_r) \tilde{\varepsilon} \\ &+ (I_b \otimes I_t \otimes I_r) \tilde{B}\tilde{\varepsilon} \\ &+ (I_b \otimes I_t \otimes I_r) \tilde{\varepsilon}\tilde{\varepsilon} \end{aligned}$$

$$\tilde{\mu} = E[\tilde{Y}] = (I_b \otimes I_t \otimes I_r) \mu_0 + (I_b \otimes I_t \otimes I_r) \tilde{\varepsilon}$$

$$\begin{aligned} \Sigma = \text{var}[\tilde{Y}] &= \sigma_B^2 (I_b \otimes I_t \otimes I_r) + \sigma_{B\tilde{\varepsilon}}^2 (I_b \otimes I_t \otimes I_r) \\ &+ \sigma_{\tilde{\varepsilon}\tilde{\varepsilon}}^2 (I_b \otimes I_t \otimes I_r) \end{aligned}$$

Show that $\Sigma = \sum_{\ell=1}^5 C_\ell A_\ell$

$$I \otimes \bar{I} \otimes \bar{I} = A_1 + A_2$$

$$I \otimes I \otimes \bar{I} = A_1 + A_2 + A_3 + A_4$$

$$I \otimes I \otimes I = A_1 + A_2 + A_3 + A_4 + A_5$$

$$\begin{aligned} \Sigma &= \text{tr } \sigma_B^2 (A_1 + A_2) + r \sigma_{B\tilde{\varepsilon}}^2 (A_1 + A_2 + A_3 + A_4) \\ &\quad + \sigma_{\tilde{\varepsilon}\tilde{\varepsilon}}^2 (A_1 + A_2 + A_3 + A_4 + A_5) \\ &= (\text{tr } \sigma_B^2 + r \sigma_{B\tilde{\varepsilon}}^2 + \sigma_{\tilde{\varepsilon}\tilde{\varepsilon}}^2) (A_1 + A_2) \end{aligned}$$

$$+ (\sigma_{B\tau}^2 + \sigma_E^2) (A_3 + A_4)$$

$$+ \sigma_E^2 A_5$$

Treatment SS

$$Y' A_3 Y \sim (\sigma_{B\tau}^2 + \sigma_E^2) \chi_{t-1}^2 (\delta_3)$$

$$\delta_3 = \frac{\mu' A_3 \mu}{\sigma_{B\tau}^2 + \sigma_E^2} = \frac{b r \sum_{j=1}^t (x_j - \bar{x})^2}{\sigma_{B\tau}^2 + \sigma_E^2} = \frac{b r \sum x_j^2}{\sigma_{B\tau}^2 + \sigma_E^2}$$

Interaction SS

$$Y' A_4 Y \sim (\sigma_{B\tau}^2 + \sigma_E^2) \chi_{(b-1)(t-1)}^2 (\delta_4)$$

$$\delta_4 = 0 \quad b/c$$

$$\begin{aligned} A_4 \mu &= (C_b \otimes C_t \otimes \bar{I}_r) \left[(I_b \otimes I_t \otimes I_r) \mu_0 \right. \\ &\quad \left. + (I_b \otimes I_t \otimes I_r) \xi \right] \\ &= 0 \quad b/c \quad C_b I_b = 0 \end{aligned}$$

Error SS

$$Y' A_5 Y = \sigma_E^2 \chi_{bt(r-1)}^2 (\delta_5)$$

$$\delta_5 = 0 \quad b/c \quad C_r I_r = 0$$

Moment Estimators

$$\hat{\sigma}_E^2 = \frac{Y' A_5 Y}{bt(r-1)} \text{ is unbiased for } \sigma_E^2$$

$$r \hat{\sigma}_{B\tau}^2 + \hat{\sigma}_E^2 = \frac{Y' A_4 Y}{(b-1)(t-1)} \text{ is } u/b \text{ for } r \sigma_{B\tau}^2 + \sigma_E^2$$

- Averaging over replicates

$$\bar{Y}_{ij\bullet} = \mu_0 + B_i + \tau_j + B\tau_{ij} + \bar{E}_{ij\bullet}$$

- Cannot separately identify σ_E^2 and $\sigma_{B\tau}^2$
- Error variance is $\sigma_{B\tau}^2 + \frac{1}{r}\sigma_E^2$
- Amatrices

$$M_0 : \bar{A}_1 = \bar{J}_b \otimes \bar{J}_t \quad \nearrow \quad \bar{A}_1 + \bar{A}_2 = I_b \otimes \bar{J}_t$$

$$B : \bar{A}_2 = C_b \otimes \bar{J}_t$$

$$\tau : \bar{A}_3 = \bar{J}_b \otimes C_t \quad \nearrow \quad \bar{A}_3 + \bar{A}_4 = I_b \otimes C_t$$

$$E : \bar{A}_4 = C_b \otimes C_t$$

$$\sum_{\ell=1}^4 \bar{A}_\ell = I_{bt} \quad \sum_{\ell=1}^4 \text{rank}(\bar{A}_\ell) = bt$$

$$\begin{aligned} \text{var} = \bar{\Sigma} &= \sigma_B^2 (I_b \otimes J_t) + (\sigma_{B\tau}^2 + \frac{1}{r}\sigma_E^2)(I_b \otimes I_t) \\ &= t\sigma_B^2 (\bar{A}_1 + \bar{A}_2) \\ &\quad + (\sigma_{B\tau}^2 + \frac{1}{r}\sigma_E^2) (\bar{A}_1 + \bar{A}_2 + \bar{A}_3 + \bar{A}_4) \end{aligned}$$

$$\text{treatment SS} \sim (\sigma_{B\tau}^2 + \frac{1}{r}\sigma_E^2) \chi_{(t-1)}^2 (\delta_3)$$

$$\text{Error SS} \sim (\sigma_{B\tau}^2 + \frac{1}{r}\sigma_E^2) \chi_{(b-1)(t-1)}^2 (0)$$

To test H_0 : no treatment effect

$$\tau_1 = \dots = \tau_t = 0 \quad \text{or} \quad \sum \tau_j^2 = 0 = \delta_3$$

$$\text{Under } H_0 \quad \frac{\text{treatment SS} / (t-1)}{\text{Error SS} / ((b-1)(t-1))} \sim F_{t-1, (b-1)(t-1)}$$

Power of the test is determined by

$$S_3 = \frac{b \sum x_j^2}{\sigma_{B\bar{x}}^2 + \sigma_E^2 / r}$$