

Corollary (Bhat's Theorem 3.4.1)

Under the conditions of Cochran's Thm.

Suppose $\gamma \sim N(\mu, \Sigma)$. Then $\Sigma = \sum_{i=1}^m c_i A_i$

where $c_i > 0$ $i=1, \dots, m$,

$$\gamma' A_i \gamma \sim c_i \chi_{n_i}^2(\delta_i)$$

independently, where $\delta_i = \mu' A_i \mu / c_i$

Proof: $\Sigma = \sum_i c_i A_i$ implies $(1/c_i) A_i \Sigma = A_i$

Hence $(1/c_i) A_i \Sigma$ is idempotent and $A_i \Sigma A_j = 0$

e.g. one-way classification

$$y_{ij} \sim N(\mu_i, \sigma^2) \text{ independently}$$

$$i=1, \dots, t, j=1, \dots, r$$

$$y' A_1 y = y' (\bar{I}_t \otimes \bar{I}_r) y = \text{tr } \bar{y}' \bar{y}$$

$$y' A_2 y = y' (C_t \otimes \bar{I}_r) y = \sum_j (\bar{y}_{j.} - \bar{y}_{..})^2$$

$$y' A_3 y = y' (I_t \otimes C_r) y = \sum_{i,j} (y_{ij} - \bar{y}_{..})^2$$

$$A_1 + A_2 + A_3 = I_{tr}$$

$$\text{rank}(A_1) = 1, \text{rank}(A_2) = t-1, \text{rank}(A_3) = t(r-1)$$

$$\Sigma = \sigma^2 (I_t \otimes I_r) = \sigma^2 \sum_{i=1}^3 A_i$$

Bhat's Lemma implies $y' A_i y \sim \chi_{n_i}^2(\delta_i)$

independent $i = 1, 2, 3$

Note that $\mu = (\mu_1, \dots, \mu_t)'$ $\otimes \frac{1}{r}$

$$\delta_1 = \text{tr} \bar{\mu} / \sigma^2 \quad \bar{\mu} = \frac{1}{t} \sum_i \mu_i$$

$$\delta_2 = \frac{\sum_{i,j} (\mu_i - \bar{\mu})^2}{\sigma^2} = r \sum_i (\mu_i - \bar{\mu})^2 / \sigma^2$$

$$\delta_3 = \sum_{i,j} (\mu_i - \mu_j)^2 / \sigma^2 = 0$$

Complete Balanced Factorial Designs

Eg. Randomized Complete Block (RCB) design with replication

1. Random sample of b homogeneous "blocks" selected from a population of potential blocks
2. Fixed set of t treatments
3. Random sample of measurements / responses from each block / treatment combination

Eg. $b = 4$, $t = 3$, $r = 2$

| 1 | 2 | 3 | 4 |
|-------|-------|---|---|
| 1 1 | 1 2 | | |
| 2 3 | 3 1 | | |
| 3 1 | 2 3 | | |

random assignment of treatments to units

$$Y_{ijk} = \text{measurement } k\text{th replicate of treatment } j \\ \text{in block } i$$

$$= \mu_0 + B_i + \tau_j + B\tau_{ij} + R(B\tau)_{(ij)k}$$

- B_i = random effect of block i

$$B_i \sim \text{iid } N(0, \sigma_B^2)$$

$$\tilde{B} = (B_1, \dots, B_b)' \sim N_b(\Omega_b, \sigma_B^2 I_b)$$

- τ_j = fixed effect of treatment j

$$\sum_{j=1}^t \tau_j = 0 \quad (\text{sum constraint})$$

- $B\tau_{ij}$ = random block/treatment interaction

$$B\tau_{ij} \sim \text{iid } N(0, \sigma_{B\tau}^2)$$

$$\tilde{B\tau} = (B\tau_{11}, \dots, B\tau_{1t}, \dots, B\tau_{b1}, \dots, B\tau_{bt})'$$

$$\sim N_{bt}(\Omega, \sigma_{B\tau}^2 I_{bt})$$

(This is called the infinite model in the book!)

- $R(B\tau)_{(ij)k}$ = k th replicate effect nested within block/treatment combination (i, j)

Note that block and treatment factors are crossed - every combination is observed.

$$\bullet R(B\varepsilon)_{(ij)k} \stackrel{\text{ind}}{\sim} N(0, \sigma_E^2)$$

" error variance
 E_{ijk}

$$\underline{E} = (E_{111}, \dots, E_{11r}, E_{121}, \dots, E_{12r}, \dots, E_{btr}, \dots, E_{btr})'$$

$$\sim N_{btr}(\Omega, \sigma_E^2 I_{btr})$$

$$\begin{aligned}\underline{Y} &= (Y_{111}, \dots, Y_{11r}, \dots, \dots, Y_{btr}, \dots, Y_{btr})' \\ &= (I_b \otimes I_t \otimes I_r) \mu_0 \\ &\quad + (I_b \otimes I_t \otimes I_r) B \\ &\quad + (I_b \otimes I_t \otimes I_r) \xi \\ &\quad + (I_b \otimes I_t \otimes I_r) B\varepsilon \\ &\quad + (I_b \otimes I_t \otimes I_r) \varepsilon\end{aligned}$$

independent multivariate normals

$$E(\underline{Y}) = (I_b \otimes I_t \otimes I_r) \mu_0 + (I_b \otimes I_t \otimes I_r) \xi$$

$$\bar{Y}_{...} = \frac{1}{btr} (I_b \otimes I_t \otimes I_r)' \underline{Y}$$

$$E(\bar{Y}_{...}) = \mu_0 + \frac{1}{t} \sum_j \tau_j = \mu_0 \text{ "overall mean"}$$

τ_j = deviation of treatment j from μ_0 .

$$\begin{aligned}\Sigma = \text{var}(\underline{Y}) &= (I_b \otimes I_t \otimes I_r) \sigma_B^2 I_b (I_b \otimes I_t \otimes I_r)' \\ &\quad + (I_b \otimes I_t \otimes I_r) \sigma_{B\varepsilon}^2 I_{bt} (I_b \otimes I_t \otimes I_r)' \\ &\quad + (I_b \otimes I_t \otimes I_r) \sigma_\varepsilon^2 I_{str} (I_b \otimes I_t \otimes I_r)'\end{aligned}$$

$$\begin{aligned}
 \text{"variance components"} &= \sigma_B^2 (I_b \otimes J_t \otimes J_r) \\
 &+ \sigma_{B\epsilon}^2 (I_b \otimes I_t \otimes J_r) \\
 &+ \sigma_\epsilon^2 (I_b \otimes I_t \otimes I_r)
 \end{aligned}$$

Sums of Squares

$$A 1. \mu_0 : btr \bar{y}_{...}^2 = y' (\bar{J}_b \otimes \bar{J}_t \otimes \bar{J}_r) y$$

$$A 2. B : \sum_{i,j,k} (\bar{y}_{ijk} - \bar{y}_{...})^2 = y' (C_b \otimes \bar{J}_t \otimes \bar{J}_r) y$$

$$A 3. T : \sum_{i,j,k} (\bar{y}_{.jk} - \bar{y}_{...})^2 = y' (\bar{J}_b \otimes C_t \otimes \bar{J}_r) y$$

$$\begin{aligned}
 A 4. B\epsilon : \sum_{i,j,k} &(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.jk} + \bar{y}_{...})^2 \\
 &= y' (C_b \otimes C_t \otimes \bar{J}_r) y
 \end{aligned}$$

$$\begin{aligned}
 A 5. R(B\epsilon) : \sum_{i,j,k} &(\bar{y}_{ijk} - \bar{y}_{ij.})^2 \\
 &= y' (I_b \otimes I_t \otimes C_r) y
 \end{aligned}$$

$$A_1 + A_2 = I_b \otimes \bar{J}_t \otimes \bar{J}_r \quad \swarrow \quad A_1 + A_2 + A_3 + A_4$$

$$A_3 + A_4 = I_b \otimes C_t \otimes \bar{J}_r \quad \swarrow = I_b \otimes I_t \otimes \bar{J}_r$$

$$\sum_1^5 A_i = I_b \otimes I_t \otimes I_r$$

$$\begin{aligned}
 \sum_{i=1}^5 \text{rank}(A_i) &= 1 + (b-1) + (t-1) + (b-1)(t-1) + btr \\
 &= btr
 \end{aligned}$$

$$\text{Show that } \Sigma = \text{var}(Y) = \sum_i c_i A_i$$