

Office hours: 4:30 - 5:30 Tuesdays

- $x_1, \dots, x_n$  independent  $N(\mu_i, 1)$  or  
 $x = (x_1, \dots, x_n)'$  ~  $N_n(\mu, I_n)$ . Then

$$x'x = \sum_i x_i^2 \sim \chi_n^2(8) \quad \begin{matrix} \sum \mu_i = \mu' \mu \\ \text{degrees of freedom} \end{matrix}$$

- If  $y \sim N_n(0, I_n)$  then  $y'Ay \sim \chi_r^2(0)$   
iff  $A^2 = A$  and  $\text{rank}(A) = r$ .

(if)  $A = PDP' = P_r D_r P_r' = P_r P_r'$

$$\text{so } y'Ay = y'P_r P_r'y = x'x = \sum_i x_i^2 \sim \chi_r^2(0)$$

$$\text{because } P_r'y \sim N(0, P_r'I_n P_r) = N_r(0, I_r)$$

(only if) Since  $y'Ay \sim \chi_r^2(0)$

$$\begin{aligned} M_{y'Ay}(t) &= (1-2t)^{-r/2} \\ &= \int_{\mathbb{R}^n} e^{ty'Ay} (2\pi)^{-n/2} e^{-\frac{1}{2}y'y} dy \\ &= \int_{\mathbb{R}^n} (2\pi)^{-n/2} e^{-\frac{1}{2}y'(I-2At)y} dy \\ &= |I-2At|^{-n/2} \end{aligned}$$

Aside: Suppose  $A$  has eigenvalue  $\lambda$   
and associated eigenvector  $x$ ; ie  $Ax = \lambda x$ .

Then

$$(i) A^k \underline{x} = \lambda^k \underline{x}$$

$$(ii) cA \underline{x} = (c\lambda) \underline{x}$$

$$(iii) (I+A) \underline{x} = (1+\lambda) \underline{x}$$

In general, if  $p(A)$  is a polynomial function of  $A$  (e.g.  $p(A) = 2A^0 + A^1 + 3A^2 = 2I + A + 3A^2$ )

then  $p(\lambda)$  is an eigenvalue of  $p(A)$  with e-vector  $\underline{x}$ .

$$\begin{aligned} (1-2t)^{-r/2} &= |I - 2At|^{-1/2} \\ &= \prod_{i=1}^n (1-2\lambda_i t)^{-1/2} \end{aligned}$$

$$\text{or } (1-2t)^r = \prod_{i=1}^n (1-2\lambda_i t)$$

Only true if  $\lambda_i = 1$  with multiplicity  $r$  and  $\lambda_i = 0$  with multiplicity 0.

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- Generalization: If  $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, I)$  then  $\mathbf{Y}' A \mathbf{Y} \sim \chi_r^2(S)$  where  $S = \boldsymbol{\mu}' A \boldsymbol{\mu}$  iff  $A^2 = A$  and  $\text{rank}(A) = r$

(if)  $\mathbf{Y}' A \mathbf{Y} = \mathbf{Y}' P_r P_r' \mathbf{Y} = \mathbf{x}' \mathbf{x} \sim \chi_r^2(\boldsymbol{\mu}' A \boldsymbol{\mu})$   
because  $\mathbf{x} \sim N(P_r \boldsymbol{\mu}, I_r)$  and  
 $\boldsymbol{\mu}' P_r P_r' \boldsymbol{\mu} = \boldsymbol{\mu}' A \boldsymbol{\mu}$

- The non-central chi-square distribution can be represented as a mixture of central chisquares. If  $\gamma \sim \chi_n^2(\delta)$

$$\gamma \sim \sum_{k=0}^{\infty} \frac{(\delta/2)^k e^{-\delta/2}}{k!} \chi_{n+2k}^2(0)$$

Aside:

- Is it true that  $AB = AB^* \Leftrightarrow B = B^*$ ?
- No! eg.  $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C \begin{pmatrix} 1 \\ 2 \end{pmatrix} = C \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  by  $\frac{1}{1} \neq \frac{1}{2}$
- If  $A$  is nonsingular then  $AB = AB^* \Leftrightarrow B = B^*$
- If " " " then  $\text{rank}(AB) = \text{rank}(B)$
- If  $A$  is symmetric, p.d., it has a symmetric square root  $A^{1/2} = P D^{1/2} P'$  with inverse  $A^{-1/2} = P D^{-1/2} P'$

Theorem: Suppose  $\gamma \sim N(\mu, \Sigma)$  where  $\Sigma > 0$

Then  $\gamma' A \gamma \sim \chi_r(\delta)$  where  $\delta = \mu' A \mu$

iff  $A \Sigma A = A$  and  $\text{rank}(A) = r$

Note: Since  $\Sigma > 0$

$$\begin{aligned}
 (i) \quad A \Sigma A = A &\Leftrightarrow \Sigma A \Sigma A = \Sigma A \\
 &\Leftrightarrow A \Sigma A \Sigma = A \Sigma \\
 &\Leftrightarrow \Sigma^{1/2} A \Sigma A \Sigma^{1/2} = \Sigma^{1/2} A \Sigma^{1/2}
 \end{aligned}$$

$$(ii) \quad \text{rank}(\Sigma A) = \text{rank}(A \Sigma) = \text{rank}(\Sigma^{1/2} A \Sigma^{1/2}) = \text{rank}(A)$$

Proof:  $X = \Sigma^{-1/2} Y$  then  $X \sim N\left(\underbrace{\Sigma^{-1/2} \mu}_{\mathbf{I}}, \underbrace{\Sigma^{-1/2} \Sigma^{-1/2}}_{\mathbf{I}}\right)$

$$\begin{aligned} Y' A Y &= Y' \Sigma^{-1/2} (\Sigma^{1/2} A \Sigma^{1/2}) \Sigma^{-1/2} Y \\ &= X' B X \\ &\sim \chi^2_r (\mu' \Sigma^{-1/2} B \Sigma^{-1/2} \mu) \quad \text{by previous result} \\ &= \chi^2_r (\mu' A \mu) \end{aligned}$$

iff  $B$  is idempotent with rank  $r$

Corollary:  $Y \sim N_n(\mu, \Sigma)$  where  $\Sigma > 0$

$$Y' \Sigma^{-1} Y \sim \chi_n(\delta), \quad \delta = \mu' \Sigma^{-1} \mu$$


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•  $Y \sim N_n(\mu, \Sigma)$ ,  $\Sigma > 0$

$A$  symmetric,  $\text{rank}(A) = r$

$$\text{then } Y' A Y \sim \sum_{i=1}^r \lambda_i z_i^2 \sim \sum_{i=1}^r \lambda_i \chi_i^2(\delta_i^*)$$

where  $z_1, \dots, z_r$  are independent  $N(\mu_i^*, 1)$

and  $\lambda_1, \dots, \lambda_p$  are the non-zero eigenvalues  
of  $A\Sigma$ .

Proof:  $Y' A Y = X' B X$  where  $B = \Sigma^{1/2} A \Sigma^{1/2}$   
and  $X = \Sigma^{-1/2} Y$ . Let  $B = Q \Delta Q'$  be the  
spectral decomposition of  $B$ . Then

$$\begin{aligned} Y' A Y &= X' B X = Z' \Delta Z \quad \text{where} \\ Z &= Q' X = Q' \Sigma^{-1/2} Y \sim N(Q' \Sigma^{-1/2} \mu, \mathbf{I}) \end{aligned}$$

because  $Q'\Sigma^{-1/2}\Sigma\Sigma^{-1/2}Q = I$ . Hence  
 $z_1, \dots, z_n \sim \text{independently } N(\mu_i^* = q_i' \Sigma^{-1/2} \mu, 1)$

where  $q_i$  is the  $i$ th col. of  $Q$ . The result follows from

$$z'\Delta z = \sum_i^n \lambda_i z_i^2$$


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Independence

- $y \sim N_n(\mu, \Sigma), \Sigma > 0$
- Recall that  $Ay$  and  $By$  are independent iff  $A\Sigma B' = 0$

Suppose  $A = A'$

- $y'Ay$  is indep of  $By$  iff  $A\Sigma B' = 0$
- If  $B' = B$  then  $y'Ay$  is indep of  $y'By$   
iff  $A\Sigma B' = A\Sigma B = 0$

Proof. Suppose  $A\Sigma B = 0$  and

$$r = \text{rank}(A) = \text{rank}(\Sigma^{1/2} A \Sigma^{-1/2}) = \text{rank}(B)$$

let  $B = Q\Lambda Q'$  where  $\lambda_1, \dots, \lambda_r$  are non-zero and  $\lambda_{r+1}, \dots, \lambda_n$  equal 0.

$$\text{So } y'Ay = z'\Delta z = \cdot$$

Similarly

$$\begin{aligned} y'By &= (y'\Sigma^{-1/2}Q)(Q'\Sigma^{1/2}B\Sigma^{1/2}Q)(Q'\Sigma^{-1/2}y) \\ &= z'Cz \end{aligned}$$

$$\begin{aligned}
 A \Sigma B = O &\iff (\Sigma^{1_2} A \Sigma^{1_2})(\Sigma^{1_2} B \Sigma^{1_2}) = O \\
 &\iff Q \Delta (Q' \Sigma^{1_2} B \Sigma^{1_2} Q) = O \\
 &\iff \Delta C = O
 \end{aligned}$$

But

$$\Delta C = \begin{pmatrix} \Delta_r & O \\ O & O \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$$\begin{aligned}
 O = A \Sigma B &\iff O = \Delta C \\
 &= \begin{pmatrix} \Delta_r C_{11} & \Delta_r C_{12} \\ O & O \end{pmatrix}
 \end{aligned}$$

implies  $C_{11}$  and  $C_{12}$  are zero, and hence also  $C_{21}$ . Hence

$$Y' A Y = Z' A Z = Z'_1 \Delta_r Z_1$$

$$\text{and } Y' B Y = Z' C Z = Z'_2 C_{22} Z_2$$

where  $Z = (Z'_1, Z'_2)'$ . Independence of  $Y' A Y$  and  $Y' B Y$  follows from the fact that  $\text{cov}(Z) = I$  and so  $Z_1$  and  $Z_2$  are indep.