

## General Linear Mixed Model

$$Y = X\beta + \sum_{i=0}^q Z_i u_i \quad u_i \sim N[0, \sigma^2 I_{c_i}]$$

- $\mathbb{I} = \sum_{i=1}^m A_i = \sum_{i=1}^{q+1} (B_i + C_i)$
- $\Sigma = \sum_{i=0}^q \sigma^2 Z_i Z_i' = \sum_{i=1}^{q+1} a_i (B_i + C_i)$
- $\sum_{i=1}^{q+1} B_i X = X \quad \sum_i B_i = X(X'X)^{-1}X'$

Full ML case

ML eqns for  $a_i$ 's reduce to

$$\begin{aligned}\hat{a}_i &= \frac{1}{p_i + r_i} (Y - X\hat{\beta})' (B_i + C_i) (Y - X\hat{\beta}) \\ &= \dots = \frac{Y' C_i Y}{p_i + r_i}\end{aligned}$$

REML case

$K \in \mathbb{R}^{n \times (n-k)}$   $\text{rank } K = n-k$ ,  $\text{rank } K(X) = k \leq p$

such that  $K'X = 0$

In particular, choose  $K$  such that

$$KK' = I - X(X'X)^{-1}X' = \sum_{i=1}^{q+1} C_i$$

and so  $KK'C_i = C_i$

$$K'Y = K'X\beta + \sum_{i=0}^q K'Z_i u_i$$

$$\text{var}(K'Y) = K' \Sigma K$$

$$= \sum \alpha_i K' C_i K$$

REML eqns for  $\alpha_i$ 's are

$$\begin{aligned}\hat{\alpha}_i &= \frac{1}{r_i} (K' Y)' K' C_i K (K' Y) \\ &= \frac{1}{r_i} Y' C_i Y \quad \text{b/c } K K' C_i = C_i\end{aligned}$$

$$\begin{aligned}E(\hat{\alpha}_i) &= \frac{1}{r_i} [ \text{tr}(C_i \Sigma) + \beta' X' C_i X \beta ] \\ &= \frac{1}{r_i} \text{tr}(\alpha_i C_i) + 0 \\ &= \alpha_i\end{aligned}$$


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General Linear Hypothesis:  $H\beta = h$

where  $H$  is  $q \times p$ ,  $\text{rank}(H) = r \leq \min(q, p)$

Also, assume  $H = MX$  so that the rows of  $H$  are linear combinations of the rows of  $X$ , so that  $H\beta$  is estimable.

The  $\hat{H\beta} = H(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y$  be the BLUE of  $H\beta$

Then

$$*(H\hat{\beta} - h)' [H(X'\Sigma^{-1}X)^{-1}H']^{-1} (H\hat{\beta} - h) \sim \chi_r^2(S)$$

$$\begin{aligned}S &= (H\beta - h)' [I - (H\beta - h)] \\ &= 0 \quad \text{if } H\beta = h\end{aligned}$$

Let  $I = \sigma_0^2 V$  where  $\sigma_0^2$  is the error variance

Then \* is

$$\frac{1}{\sigma_0^2} (H\hat{\beta} - h)' [H(X'V^{-1}X)^{-1}H']^{-1} (H\hat{\beta} - h)$$

In practice the variance components

$\sigma_0^2, \dots, \sigma_q^2$  are replaced by estimates

Define

$$F = \frac{1}{n} (H\hat{\beta} - h)' [H(X'V^{-1}X)^{-1}H']^{-1} (H\hat{\beta} - h) / \hat{\sigma}_0^2$$

$$= \frac{1}{n} GSS(H) / \hat{\sigma}_0^2$$

$GSS(H)$  = generalized SS associated  
with the hypothesis  $H\beta = h$

$F$  = generalized F-statistic

Complete Balanced Designs

$$I = \sum A_i = \sum_{i=1}^{q+1} (B_i + C_i)$$

$$\Sigma = \sum_{i=1}^{q+1} a_i (B_i + C_i)$$

$$\sum_i B_i X = X \quad \sum_i C_i = X(X'X)^{-1}X'$$

Recall :  $\hat{\beta} = (X'\Sigma X)^{-1} X' \Sigma^{-1} Y = (X'X)^{-1} X' Y$

$$\text{var}(\hat{\beta}) = (X'\Sigma X)^{-1} = (X'X)^{-1} X' \Sigma X (X'X)^{-1}$$

Consider  $\gamma' B_i y \sim \alpha_i \chi_{p_i}^2(\delta_i)$

where  $\delta_i = \beta' x' B_i x \beta / \alpha_i$

$\delta_i = 0$  if  $B_i x \beta = 0$

Then

$$\begin{aligned} & (B_i x \hat{\beta})' [B_i x (x' x)^{-1} (x' x) (x' x)^{-1} x' B_i]^{-1} B_i x \hat{\beta} \\ &= \gamma' B_i [B_i x (x' x)^{-1} (x' x) (x' x)^{-1} x' B_i]^{-1} B_i y \\ &\text{if } B_i x \hat{\beta} = B_i x (x' x)^{-1} x' y = B_i y \\ &= \gamma' B_i [B_i \sum B_i]^{-1} B_i y \\ &= \gamma' B_i (\alpha_i B_i)^{-1} B_i y \\ &= \alpha_i^{-1} \gamma' B_i y \end{aligned}$$

$\gamma' B_i y \sim \alpha_i \chi_{p_i}^2(\delta_i)$

$\gamma' C_i y \sim \alpha_i \chi_{r_i}^2(0)$

$$F = \frac{\gamma' B_i y / p_i}{\gamma' C_i y / r_i} \sim F_{p_i, r_i}(\delta_i)$$

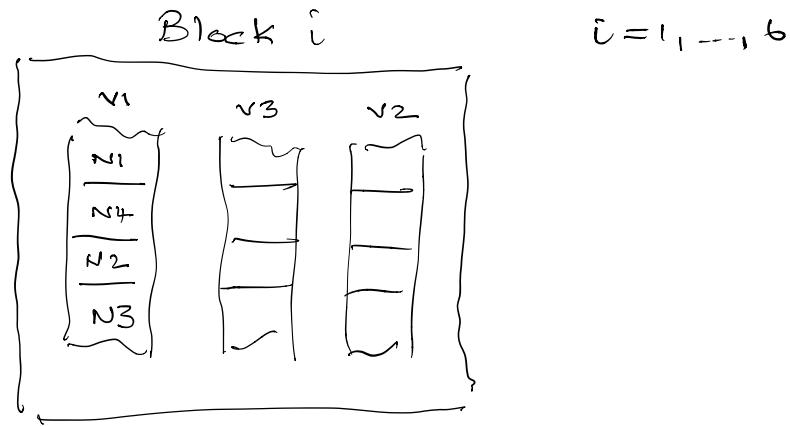
$$= \frac{1}{p_i} (B_i x \hat{\beta})' [B_i x (x' x)^{-1} x' B_i]^{-1} B_i x \hat{\beta} / \hat{\sigma}_i^2$$

$$\text{Or } \gamma' B_i y = \frac{\hat{\alpha}_i}{\hat{\sigma}_i^2} \text{GSS}(B_i x)$$

$$\text{Or } \text{GSS}(B_i x) = \frac{\hat{\sigma}_i^2}{\hat{\alpha}_i} \gamma' B_i y$$

## Split-plot experiment

- 18 whole plots arranged in six blocks of 3
- 3 varieties, 1 assigned to each whole plot in each block
- 4 nitrogen levels, 1 to each split plot in each whole plot



$$Y_{ijk} = \mu + B_i + V_j + BV_{ij} + N_k + BN_{ik} \\ + VN_{ijk} + BVN_{ijk}$$

Model term	EMS
B	$12\sigma_B^2 + 4\sigma_{BV}^2 + 3\sigma_{BN}^2 + \sigma_E^2$
V	$4\sigma_{BV}^2 + \sigma_E^2 + S_V$
BV	$4\sigma_{BV}^2 + \sigma_E^2$
N	$3\sigma_{BN}^2 + \sigma_E^2 + S_N$
BN	$3\sigma_{BN}^2 + \sigma_E^2$
VN	$\sigma_E^2 + S_{VN}$

### ANOVA table

Fixed Effects	Nom DF	Denom DF	F-ratio
V	2	10	$MSV/MSBV$
N	3	15	$MSN/MSBN$
VN	6	30	$MSVN/MSE$

EMS Estimates

EMS REML

$$MSBV = 4 \hat{\sigma}_{BV}^2 + \hat{\sigma}_E^2$$

$$\hat{\sigma}_{BV}^2 = 98.8 \quad 106.1$$

$$MSBN = 4 \hat{\sigma}_{BN}^2 + \hat{\sigma}_E^2$$

$$\hat{\sigma}_{BN}^2 = -28.9 \quad 0$$

$$MSE = \hat{\sigma}_E^2$$

$$\hat{\sigma}_E^2 = 206.0 \quad 177.1$$