

## Balanced Block Design

$t$  treatments

$K \leq t$  trts. in each block

each trt. occurs in  $r$  blocks

$b$  blocks

$$(1) \quad bK = tr$$

Balanced if  $\lambda = r \frac{K-1}{t-1}$

$\lambda$  = number times each pair of trts.  
occur together in a block

$$Y_{ij} = \mu + B_i + \gamma_j + \epsilon_{ij} \\ i = 1, \dots, b, \quad j \in T_i$$

$$B_i \sim N(0, \sigma_B^2) \quad \epsilon_{ij} \sim N(0, \sigma_\epsilon^2 = \sigma_{BT}^2)$$

$$\text{sum constraints} \quad \sum_i \gamma_j = 0$$

$i$ th block

$$Y_i = \frac{1}{K} \mu + \frac{1}{K} B_i + \gamma_i + \epsilon_i$$

Helmert transformation,  $H_K = [h_1, \dots, h_K]$ ,  
 $K \times K$  Helmert matrix

$$Y_{ij}^* = h_j' Y_i = h_j' \frac{1}{K} \mu + h_j' \frac{1}{K} B_i + h_j' \gamma_i + \epsilon_{ij}^*$$

$$\text{Note: } \text{var}(\epsilon_{ij}^*) = \sigma_{BT}^2 h_j' I_K h_j = \sigma_{BT}^2$$

Inter-block model

$$\begin{aligned} Y_i^* &= b_i' \bar{Y}_i = \sqrt{k} \mu + \sqrt{k} B_i + b_i' \bar{Z}_i + E_{ii}^* \\ &= \sqrt{k} \mu + \frac{1}{\sqrt{k}} \sum_k Z_{ik} + (\sqrt{k} B_i + E_{ii}^*) \end{aligned}$$

In the complete blocks model  $\frac{1}{\sqrt{k}} \sum_k Z_{ik} = \frac{1}{n} \bar{Z} = 0$

No information about treatment differences

$$\text{Notice that } \text{var}(\sqrt{k} B_i + E_{ii}^*) = k \sigma_B^2 + \sigma_{BT}^2$$

Intra-block model

$$Y_{ij}^* = b_{ij}' \bar{Y}_i = b_{ij}' \bar{Z}_{ij} + E_{ij}^*, \quad \begin{array}{l} i=1, \dots, b \\ j \in T_i \end{array}$$

$$\text{var}(E_{ij}^*) = \sigma_{BT}^2$$


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Intra-block and Inter-block Treatment

Difference Estimators

$V_j$  = total of all responses to trt.  $j$

$T_j$  = total of all responses in blocks containing trt.  $j$

$$E(V_j) = r \mu_j \quad \text{where } \mu_j = \mu + \tau_j$$

$$E(T_j) = r \mu_j + \lambda \sum_{i \neq j} \mu_i$$

$$= r \mu_j + \lambda \mu_j + \lambda \sum_{i \neq j} \mu_i$$

$$E\left(\frac{V_j - V_{j'}}{r}\right) = \pi_j - \pi_{j'}$$

$$\begin{aligned} E(T_j - T_{j'}) &= r(\mu_j - \mu_{j'}) + \lambda(\mu_{j'} - \mu_j) \\ &= (r - \lambda)(\pi_j - \pi_{j'}) \end{aligned}$$

$$E\left(\frac{T_j - T_{j'}}{r - \lambda}\right) = \pi_j - \pi_{j'}$$

Note  $\frac{\kappa(t-1)}{t(\kappa-1)} - \frac{r-\lambda}{t\lambda} = 1$

Define

$$\begin{aligned} \hat{\Theta}_1 &= \frac{\kappa(t-1)}{t(\kappa-1)} \stackrel{?}{=} (V_j - V_{j'}) - \frac{r-\lambda}{t\lambda} \frac{1}{r-\lambda} (T_j - T_{j'}) \\ &= \frac{\kappa}{t\lambda} (Q_j - Q_{j'}) \end{aligned}$$

where  $Q_j = V_j - \frac{1}{\kappa} T_j$

$$\hat{\Theta}_2 = \frac{1}{r-\lambda} (T_j - T_{j'})$$

Example treatments

	1	2	3	4
1	x	x	x	
2	x	x		x
3	x		x	x
4		x	x	x

blocks

$$t=4 \quad K=3 \quad b=4 \quad \lambda = 2$$

$$V_1 = \gamma_{11} + \gamma_{21} + \gamma_{31}$$

$$V_2 = \gamma_{12} + \gamma_{22} + \gamma_{42}$$

$$T_1 = \text{blocks } 1, 2, 3$$

$$T_2 = \text{block } 1, 2, 4$$

$$V_1 - V_2 = (\gamma_{11} + \gamma_{21} + \gamma_{31}) - (\gamma_{12} + \gamma_{22} + \gamma_{42})$$

$$T_1 - T_2 = (\gamma_{31} + \gamma_{33} + \gamma_{34}) - (\gamma_{42} + \gamma_{43} + \gamma_{44})$$

$$Q_1 - Q_2 = V_1 - V_2 - \frac{1}{3}(T_1 - T_2)$$

$$= (\gamma_{11} - \gamma_{12}) + (\gamma_{21} - \gamma_{22})$$

$$+ \frac{1}{3}(2\gamma_{31} - \gamma_{33} - \gamma_{34})$$

$$- \frac{1}{3}(2\gamma_{42} - \gamma_{43} - \gamma_{44})$$

- $\hat{\theta}_1$  and  $\hat{\theta}_2$  are independent unbiased estimates of  $\tau_j - \tau_{j'}$

- The BLUE of  $\tau_j - \tau_{j'}$  is

$$\hat{\theta} = \frac{\hat{\theta}_1 / \text{var}(\hat{\theta}_1) + \hat{\theta}_2 / \text{var}(\hat{\theta}_2)}{1 / \text{var}(\hat{\theta}_1) + 1 / \text{var}(\hat{\theta}_2)}$$

$$\text{var}(\hat{\theta}_1) = \frac{2K}{t\lambda} \sigma_{BT}^2 \quad \text{var}(\hat{\theta}_2) = \frac{2K(t-1)}{n(t-K)} (\kappa \sigma_B^2 + \sigma_{BT}^2)$$

## Less than Full Rank Models

Single k-level factor

$\underline{x}_j$  =  $n \times 1$  vector indicating which obs. are from level  $j$

i<sup>th</sup> component of  $\underline{x}_j$  =  $\begin{cases} 1 & \text{i<sup>th</sup> obs. is from level } j \\ 0 & \text{otherwise} \end{cases}$

Consider the X-matrix

$$X = [1_n, \underline{x}_1, \dots, \underline{x}_k]$$

X is not full rank because  $\sum_{j=1}^k \underline{x}_j = 1_n$

### Solutions

- ① Drop  $\underline{x}_k$  — this makes the last level the reference category (SAS)
- ② Drop  $\underline{x}_1$  — this makes the first level the reference (E)
- ③ Drop  $\underline{x}_k$ , but change  $\underline{x}_j$  to  $\underline{x}_j - \underline{x}_k$   
effects coding

$$Y = [1_n, \underline{x}_1 - \underline{x}_k, \dots, \underline{x}_{k-1} - \underline{x}_k] \beta + E$$

$$E(Y_j) = \beta_0 + \beta_j \quad \text{if i<sup>th</sup> obs. is from level } j \\ j = 1, \dots, k-1$$

$$E(Y_i) = \beta_0 - \sum_{j=1}^{k-1} \beta_j \quad \text{if } i \text{th obs. from level } k$$

$$\mu_j = \beta_0 + \beta_j \quad j=1, \dots, k-1$$

$$\mu_k = \beta_0 - \sum_{j=1}^{k-1} \beta_j$$

$$\bar{\mu} = \frac{1}{k} \sum_j \mu_j = \beta_0$$

$$\beta_j = \mu_j - \bar{\mu}$$


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Some useful matrix results

① (a)  $A'A = 0$  implies  $A = 0$

(b)  $\text{tr}(A'A) = 0$  implies  $A = 0$

Proof:  $\text{tr}(A'A) = \sum_{i,j} a_{ij}^2$

②  $PXX' = QXX'$  implies  $PX = QX$

Proof:  $(PXX' - QXX')(P - Q)' = (PX - QX)(PX - QX)'$

Hence the result follows from ①(a)

③ Suppose  $G$  is g-inverse of  $X'X$   
(i.e.  $X'XG X'X = X'X$ )

(a)  $G'$  is a g-inverse of  $X'X$

(b)  $XG X'X = X$

(c)  $XGx'$  is symmetric and invariant to the choice of  $G$ .

Proof

(a)  $x'x = x'xGx'x$   
 $x'x = (x'xGx'x)' = x'xG'x'x$

(b)  $x'xG'x'x = x'x$  is of the form  
 $Px'x = Qx'x$  where  $P = x'xG'$   
and  $Q = I$ . Then it follows from (2)  
that  $Px' = x'$  or  $xP' = x$   
 $XGx'x = x$

(c) Suppose  $F$  and  $G$  are g-inverses  
of  $x'x$ . Then by (b)

$$x = xFx'x = xGx'x$$

Hence by (2)  $xFx' = xGx'$

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Consider the linear model

$$y = x\beta + e \quad e \sim N(0, \sigma^2 I_n)$$

The least squares equations are

$$x'x\beta = x'y$$

Suppose  $x$  is  $n \times p$  with rank  $k \leq p$

Then  $\text{rank}(X'X) = K$  and so the equations do not have unique solution

Let  $G$  be a g-inverse of  $X'X$ , and define

$$\tilde{\beta} = GX'y + [GX'x - I]z$$

where  $z \in R^P$  is arbitrary. Then

$$\begin{aligned} X'X\tilde{\beta} &= X'XGX'y + [X'XGX'x - X'x]z \\ &= X'XGX'y + 0 \\ &= X'y \quad \text{by 3(b)} \end{aligned}$$