

$$\bar{T} = \frac{Y}{\sqrt{X/K}} \quad Y \sim N(\mu, 1) \quad X \sim \chi^2_K(0)$$

non-centrality = μ

$$F = T^2 \sim F_{1, K}(\mu^2)$$

$$Y_1, \dots, Y_n \sim N(\mu, \sigma^2)$$

$$\bar{Y} \sim N(\mu, \frac{\sigma^2}{n}) \quad S^2 \sim \frac{\sigma^2}{n-1} \chi^2_{n-1}(0)$$

$$\frac{\bar{Y}}{S/\sqrt{n}} = \frac{\bar{Y}/\sigma/\sqrt{n}}{(S/\sqrt{n})/(\sigma/\sqrt{n})} = \frac{N(\sqrt{n}\mu/\sigma, 1)}{S/\sigma}$$

non-centrality is $\sqrt{n}\mu/\sigma$

$$\text{Paired t-test} \quad \lambda = \frac{\mu_1 - \mu_2}{\sqrt{2(1-\rho)\sigma^2/n}}$$

Independent samples t-test

$$\lambda = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

full factorial model

i subject random

j drug fixed

k time fixed

every combination of factors is observed
one time - 3 factors are crossed

$$\gamma_{ijk} = \mu + S_i + D_j + SD_{ij} + T_k + ST_{ik} + DT_{jk} + E_{ijk}$$

Alternative design

In subjects, n assigned to drug A
 n " " " " B

Each subject measured at t timepoints

- subjects nested in drug
- drug and time are crossed

$$\gamma_{ijk} = \mu + SD_{ij} + D_j + T_k + DT_{jk} + E_{ijk}$$

General linear Hypothesis

$$Y = X\beta + E, \quad E \sim N(0, \sigma^2 I)$$

$$H_0: H\beta = h \quad H_{q \times p}$$

$$H\hat{\beta} - h \sim N[H\beta - h, H(X'X)^{-1}H'\sigma^2]$$

$$(H\hat{\beta} - h)' [\sigma^2 H(X'X)^{-1} H']^{-1} (H\hat{\beta} - h)$$

$$F = \frac{(H\hat{\beta} - h)' [H(X'X)^{-1} H']^{-1} (H\hat{\beta} - h) / q}{n \hat{\sigma}^2 / (n - p)}$$

Confidence region for linear combinations
of β

$$F(\beta) = \frac{(\hat{\beta} - \beta)' H' [H(X'X)^{-1} H']^{-1} H (\hat{\beta} - \beta)}{n \hat{\sigma}^2 / (n-p)}$$

$$\sim F_{q, n-p}(0)$$

A $100(1-\alpha)\%$ confidence ellipsoid
for $H\beta$ is given by

$$\left\{ \beta : F(\beta) < F_{q, n-p, 1-\alpha} \right\}$$

$$\text{eg. } H = I_p$$

$$\left\{ \beta : (\hat{\beta} - \beta)' X'X (\hat{\beta} - \beta) < \frac{n}{n-p} \hat{\sigma}^2 F_{p, n-p, 1-\alpha} \right\}$$

$$\text{or } H = g'$$

$$\left\{ \beta : \frac{(g'\hat{\beta} - g'\beta)^2}{g'(X'X)^{-1} g} < \frac{n}{n-p} \hat{\sigma}^2 F_{1, n-p, 1-\alpha} \right\}$$

Chapter 7 : Unbalanced and Missing Data

- X_d $K \times p$ — K distinct rows of X
- i th row occurs r_i times
- replication matrix

$$R = \text{blockdiag}(I_{r_i}) = \begin{pmatrix} I_{r_1} & & & \\ & I_{r_2} & 0 & \\ & & \ddots & \\ 0 & & & I_{r_k} \end{pmatrix}$$

Then $X = RX_d$

$$\hat{\beta} = (X'X)^{-1}X'\gamma = (X_d'R'R X_d)^{-1}X_d'R'\gamma$$

$$= (X_d'D X_d)^{-1}X_d'D\bar{\gamma}$$

where $D = R'R = \text{diag}(r_i)$

and $\bar{\gamma} = D^{-1}R'\gamma$

The pure error sum of squares

$$\gamma' A_{pe} \gamma = \gamma' \text{blockdiag}(C_{r_i}) \gamma$$

$$= \gamma' [I_n - RD^{-1}R'] \gamma$$

ANOVA table

Source	df	SS
Overall mean	1	$\gamma' \bar{Y}_n \gamma$
Model/Regression	p-1	$\gamma' H \gamma$
Lack-of-fit	k-p	$\gamma' [RD^{-1}R' - H] \gamma$
Pure error	n-k	$\gamma' [I - RD^{-1}R'] \gamma$

The regression SS can be further partitioned

$$X = [X_1, X_2, \dots, X_m] \quad X_1 = 1$$

$$= R[X_{1d}, X_{2d}, \dots, X_{md}]$$

Pattern Matrix and Missing Data

- Complete data model $\mathbf{Y}^* = \mathbf{X}^* \boldsymbol{\beta} + \mathbf{E}^*$
 $n \times 1 \quad n \times p \quad p \times 1 \quad n \times 1$
- Suppose obs. (i_1, \dots, i_m) are missing
- Let e_i to be the unit vector with 1 in position i
- Define the pattern matrix M with rows e_i' for $i \in \{1, 2, \dots, n\} - \{i_1, \dots, i_m\}$
- If $\mathbf{Y}^* \sim N_n[\mathbf{X}^* \boldsymbol{\beta}, \Sigma^*]$
 then $\mathbf{Y} = M \mathbf{Y}^* \sim N_{n-m}[\mathbf{X} \boldsymbol{\beta}, \Sigma]$
 where $\mathbf{X} = M \mathbf{X}^*$ and $\Sigma = M \Sigma^* M'$

Balanced Incomplete Block Design

Consider a RCB Design with b blocks and t treatment each occurring once in each block

$$\mathbf{Y}^* = \mathbf{X}^* \boldsymbol{\mu} + \mathbf{E}$$

$$\mathbf{Y}^* = (\gamma_{11}, \dots, \gamma_{1t}, \dots, \gamma_{b1}, \dots, \gamma_{bt})'$$

$$\mathbf{X}^* = \mathbf{1}_b \otimes \mathbf{I}_t$$

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_t)'$$

$$E^* \sim N_{bt}(0, \Sigma^*)$$

$$\Sigma^* = \sigma_B^2(I_b \otimes I_t) + \sigma_{BT}^2(I_b \otimes I_t)$$

Now, consider a BIBD with $k \leq t$
observations per block

- Each treatment occurs the same number of times denoted by

$$r = \frac{bk}{t}$$

$$\text{eg. } t=4, k=2, r = \frac{b}{2}$$

b must be a multiple of 2

- Each pair of treatments together in the same block an equal number of times

$$\lambda = r \times \frac{k-1}{t-1}$$

↑ ↗
 # times treatment fraction of times
 i occurs treatment i occurs
 with treatment j

$$\text{eg. } t=4, k=2$$

$$\lambda = \frac{r}{3} = \frac{b}{6}$$

b must be a multiple of 6

Minimised $B_i B D$ is

$$\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{1}} \quad \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\text{3}} \quad \underbrace{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}_{\text{4}} \quad \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\text{3}} \quad \underbrace{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{\text{4}} \quad \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\text{4}}$$

Let M denote the $b_K \times b_T$ pattern matrix such that $Y = M Y^*$ and

$$Y = X \mu + E$$

where $X = M X^*$ and $E \sim N_{b_K}(0, M \Sigma^* M')$

$$\Sigma = M \Sigma^* M'$$

$$= \sigma_B^2 M (I_b \otimes I_t) M' + \sigma_{BT}^2 M (I_b \otimes I_t) M'$$

Note that $M = \text{blockdiag}(M_i)$

where M_i is $K \times T$, such $M_i M_i' = I_K$