

$$Y = X\beta + \epsilon$$

$$\text{Model / Regression SS} \quad y' x (x' x)^{-1} x' y$$

$$X = [x_1, x_2, \dots, x_m]$$

$$\text{if } x_i' x_j = 0 \quad i \neq j$$

$$B_i = x_i (x_i' x_i)^{-1} x_i' \quad \sum_{i=1}^m B_i = x (x' x)^{-1} x'$$

$$B_i x_i = x_i \quad \sum_i B_i x = x$$

Suppose (1) $I_n = \sum_i A_i = \sum_i (B_i + C_i)$

(2) $\Sigma = \sum_i \alpha_i (B_i + C_i)$

(3) $\sum_i B_i x = x$

Then $\hat{\beta} = (x' x)^{-1} x' y = (x' \Sigma^{-1} x)^{-1} x' \Sigma^{-1} y$

$$\begin{aligned} \text{var}(\hat{\beta}) &= (x' x)^{-1} x' \Sigma x (x' x)^{-1} \\ &= (x' \Sigma^{-1} x)^{-1} \end{aligned}$$

$B_i = Q_i Q_i'$, $Q_i \ n \times p_i$, $Q_i' Q_i = I_{p_i}$

$\sum_i B_i = \sum_i Q_i Q_i' = Q Q'$, $Q = [Q_1, Q_2, \dots, Q_{q+1}]$

$Q \ n \times p$, $Q' Q = I_p$

$$Q Q' x = x \Rightarrow Q = x (Q' x)^{-1}$$

$$Q' = (x' Q)^{-1} x'$$

$$\begin{aligned} X' \Sigma^{-1} X &= X' \sum_i \alpha_i^* (B_i + C_i) X \\ &= X' \sum_i \alpha_i^* B_i X \\ &= X' Q A^{-1} Q' X \end{aligned}$$

$$\begin{aligned} (X' \Sigma^{-1} X)^{-1} &= (Q' X)^{-1} A (X' Q)^{-1} \\ &= [(X' Q)^{-1} X' X]^{-1} A [X' X (Q' X)^{-1}]^{-1} \\ &= (X' X)^{-1} X' Q A Q' X (X' X)^{-1} \\ &= (X' X)^{-1} X' I X (X' X)^{-1} \end{aligned}$$

The General Linear Hypothesis

$$y \sim N(X\beta, \sigma^2 I)$$

$$H_0: H\beta = h \quad \text{where } H \in \mathbb{R}^{q \times p}, \text{rank} = q$$

and h is $q \times 1$

$$\begin{aligned} H\hat{\beta} - h &\sim N\left[H\beta - h, \sigma^2 H(X'X)^{-1} H'\right] \\ \Rightarrow (H\hat{\beta} - h)' \left[H(X'X)^{-1} H' \right]^{-1} (H\hat{\beta} - h) &\sim \sigma^2 \chi_q^2 \end{aligned}$$

$$y \sim N(X\beta, \sigma^2 V)$$

$$\Rightarrow V^{-1/2} y \sim N\left[V^{-1/2} X\beta, \sigma^2 I_n\right]$$

e.g. One-way classification

$t+1$ treatments, with r replicates of each

$$Y_{ij} = \mu_i + \varepsilon_{ij}$$

$$\text{or } Y = X\beta + \varepsilon \quad X = I_{t+1} \otimes \mathbf{1}_r$$

H_0 : no difference between treatments
excluding the control

$$t \times (t+1) \begin{Bmatrix} 0 & 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & -1 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & -1 & \dots & -1 \end{Bmatrix} \beta = 0$$

H_0 : Average of treatments same as the control

$$\mu_0 = \frac{1}{t} \sum_i^t \mu_i$$

$$r \times (t+1) \quad \frac{1}{t} [t \ -1 \ -1 \ -1 \ \dots \ -1] \beta = 0$$

let G be $(p-q) \times p$ such that $\begin{bmatrix} H \\ G \end{bmatrix} \quad p \times p$
is full rank, and $G H' = 0$. write

$$\begin{matrix} q \\ p-q \end{matrix} \begin{bmatrix} H \\ G \end{bmatrix}^{-1} = \begin{bmatrix} R & S \end{bmatrix}_{p \times q \quad p \times (p-q)}$$

Consider the reparameterization

$$\begin{bmatrix} H \\ G \end{bmatrix} \beta = \begin{bmatrix} h \\ \theta \end{bmatrix}$$

H_0 specifies the first q components of the new parameter vector

$$\begin{aligned} Y &= X\beta + E \\ &= X(RS)\begin{bmatrix} H \\ G \end{bmatrix}\beta + E \\ &= XRH\beta + XS\theta + E \\ &= XRh + XS\theta + E \end{aligned}$$

The likelihood based on $Y \sim N(X\beta, \sigma^2 I)$

$$L = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}(Y-X\beta)'(Y-X\beta)\right\}$$

The likelihood-ratio for testing H_0 is

$$\Lambda = \frac{\max_{\beta, \sigma^2, H\beta=h} L(\beta, \sigma^2)}{\max_{\beta, \sigma^2} L(\beta, \sigma^2)}$$

Reject H_0 for small values of Λ .

- o Unrestricted case

$$\hat{\beta} = (X'X)^{-1}X'Y \quad \hat{\sigma}^2 = \frac{1}{n} Y' [I - X(X'X)^{-1}X'] Y$$

$$= \frac{1}{n} (\gamma - X\hat{\beta})' (\gamma - X\hat{\beta})$$

$$L(\hat{\beta}, \hat{\sigma}^2) = (2\pi \hat{\sigma}^2)^{-n/2} e^{-n/2}$$

- Restricted case $H\beta = h$

$$\varepsilon = \gamma - Xh$$

Estimates have the usual form with γ replaced by ε and X by X_S .

Call the estimate of σ^2 , $\tilde{\sigma}^2$

$$L(\tilde{\beta}, \tilde{\sigma}^2) = (2\pi \tilde{\sigma}^2)^{-n/2} e^{-n/2}$$

- Hence $\Lambda = (\tilde{\sigma}^2 / \hat{\sigma}^2)^{-n/2}$

- Define $F = (\Lambda^{-2/n} - 1) \frac{n-p}{q}$
 $= \frac{(\tilde{\sigma}^2 - \hat{\sigma}^2) / q}{\hat{\sigma}^2 / n-p}$

Reject H_0 for large values of F

- $(\gamma - X\tilde{\beta})' (\gamma - X\tilde{\beta}) = (\gamma - X\hat{\beta})' (\gamma - X\hat{\beta}) + (\hat{\beta} - \tilde{\beta})' X' X (\hat{\beta} - \tilde{\beta})$

Cross term is zero b/c $X'(\gamma - X\hat{\beta}) = 0$

It follows that

$$F = \frac{(\hat{\beta} - \tilde{\beta})' X' X (\hat{\beta} - \tilde{\beta}) / a}{Y' [I - X(X'X)^{-1}X'] Y / (n-p)}$$

Derivation of $\tilde{\beta}$ using Lagrange multipliers

$$\max_{\beta, \sigma^2, H\beta = h} \ell(\beta, \sigma^2) = \max_{\beta, \sigma^2, \lambda} \left[\ell(\beta, \sigma^2) - \lambda' [H\beta - h] \right]$$

Take derivatives of

$$\ell(\beta, \sigma^2) - \lambda' (H\beta - h)$$

with respect to β , σ^2 and λ .

$$\beta \quad (1) \quad \frac{1}{\sigma^2} X' (Y - X\beta) - H'\lambda = 0$$

$$\sigma^2 \quad (2) \quad -\frac{n}{2\sigma^2} - \frac{1}{2\sigma^2} (Y - X\beta)' (Y - X\beta) = 0$$

$$\lambda \quad (3) \quad H\beta - h = 0$$

$$Eq. (2) \rightarrow \tilde{\sigma}^2 = \frac{1}{n} (Y - X\tilde{\beta})' (Y - X\tilde{\beta})$$

$$Eq. (1) \rightarrow -X'X\tilde{\beta} + X'Y = \tilde{\sigma}^2 H'\lambda$$

$$\tilde{\beta} = \hat{\beta} - (X'X)^{-1} H'\tilde{\lambda}^* \quad \tilde{\lambda}^* = \tilde{\lambda}\tilde{\sigma}^2$$

$$Eq. (3) \rightarrow H\tilde{\beta} - h = H(X'X)^{-1} H'\tilde{\lambda}^*$$

$$\tilde{\lambda}^* = [H(x'x)^{-1}H']^{-1}(H\hat{\beta} - h)$$

Hence $\tilde{\beta} = \hat{\beta} - (x'x)^{-1}H'[H(x'x)^{-1}H']^{-1}(H\hat{\beta} - h)$

Finally this implies

$$F = \frac{(H\hat{\beta} - h)'[H(x'x)^{-1}H']^{-1}(H\hat{\beta} - h)/\sigma^2}{y'[I - x(x'x)^{-1}x']y/(n-p)}$$

Note that

$$y'[I - x(x'x)^{-1}x']y \sim \sigma^2 \chi_{n-p}^2(0)$$

independently of

$$H\hat{\beta} - h \sim N[H\beta - h, \sigma^2 H(x'x)^{-1}H']$$

$$\text{So } F \sim F_{q, n-p}(\delta)$$

where $\delta = (H\beta - h)'[H(x'x)^{-1}H']^{-1}(H\beta - h)/\sigma^2$

$$\text{So } \delta = 0 \text{ if } H\beta = h$$