

Variance components model

$$Y = X\beta + \sum_{i=1}^q Z_i u_i + \varepsilon$$

$$u_i \sim N_{c_i}(0, \sigma_i^2 I_{c_i}) \quad i=1, \dots, q$$

$$\varepsilon \sim N_n(0, \sigma^2 I_n)$$

$$Y = X\beta + \sum_{i=0}^q Z_i u_i \quad u_0 = \varepsilon \\ \sigma_u^2 = \sigma^2$$

$$Y \sim N_n \left[ \mu = X\beta, \Sigma = \sum_{i=0}^q \sigma_i^2 Z_i Z_i' \right]$$

ML equations

$$(1) \quad \hat{\beta} = (X' \Sigma^{-1} X)^{-1} X' \Sigma^{-1} Y$$

$$(2) \quad \text{tr}(\Sigma^{-1} Z_i Z_i') = Y' P Z_i Z_i' P Y$$

$$P = \Sigma^{-1} - \sum_{i=0}^q X (X' \Sigma^{-1} X)^{-1} X' \Sigma^{-1}$$

RCB Design

$$Y_{ijk} = \mu_0 + B_i + T_j + BT_{ij} + E_{ijk}$$

$$Y = X\beta + E \quad X = I_b \otimes I_t \otimes I_r$$

$$A_1 = \bar{J}_b \otimes \bar{J}_c \otimes \bar{J}_r$$

$$A_2 = C \otimes \bar{J} \otimes \bar{J}$$

$$A_3 = \bar{J} \otimes C \otimes \bar{J}$$

$$A_4 = C \otimes C \otimes \bar{J}$$

$$A_5 = I \otimes I \otimes C$$

$$A_1 + A_3 = \bar{J} \otimes I \otimes \bar{J} = X(X'X)^{-1}X'$$

$$\begin{aligned} A_2 + A_4 + A_5 &= I \otimes I \otimes I - \bar{J} \otimes I \otimes \bar{J} \\ &= I - X(X'X)^{-1}X' \end{aligned}$$

$$\Sigma = \left( r b \sigma_B^2 + r \sigma_{BT}^2 + \sigma^2 \right) (A_1 + A_2)$$

$$+ \left( r \sigma_{BT}^2 + \sigma^2 \right) (A_3 + A_4)$$

$$+ \sigma^2 A_5$$

$$= \alpha_1 (B_1 + C_1) + \alpha_2 (B_2 + C_2)$$

$$+ \alpha_3 (B_3 + C_3) \quad \text{where } B_3 = 0$$

Note that  $\sum_{i=1}^3 B_i X = X$

Suppose we can rewrite the ANOVA decomposition as follows

$$I = \sum_{i=1}^m A_i = \sum_{i=1}^{q+1} (B_i + C_i) \quad (1)$$

where  $B_i$  is either a zero matrix or an A-matrix associated with a fixed effect, and  $C_i$  is an A-matrix associated with a random effect.

- Let  $p_i = \text{rank}(B_i) \geq 0$

and  $r_i = \text{rank}(C_i)$

$$\sum_{i=1}^{q+1} p_i = P \quad \sum_{i=1}^{q+1} (p_i + r_i) = n$$

- Suppose in addition that

$$\Sigma = \sum_{i=1}^{q+1} a_i (B_i + C_i) \quad (2)$$

for some fixed constants  $a_1, \dots, a_{q+1}$

- Finally, suppose that

$$\sum_{i=1}^{q+1} B_i X = X \quad (3)$$

Note that  $B_i B_j = 0$ ,  $B_i C_j = 0$  and  $C_i C_j = 0$  for  $i \neq j$ . This implies

$$\Sigma^{-1} = \sum_{i=1}^{q+1} a_i^{-1} (B_i + C_i)$$

Theorem If conditions 1-3 hold, then the solutions of the ML equations are

$$\hat{\beta} = (X'X)^{-1} X'y$$

$$\hat{Q}_i = \frac{Y'C_i Y}{\hat{\beta}_i + r_i}$$

Proof. Note that  $C_j X = C_j \sum_{i=1}^{q+1} B_i X = 0$

Define  $Q = [Q_1, Q_2, \dots, Q_{q+1}]$

where  $Q_i$  is  $n \times p_i$  and  $B_i = Q_i Q_i'$

$$\text{Then } Q Q' = \sum_{i=1}^{q+1} Q_i Q_i' = \sum_{i=1}^{q+1} B_i$$

$$\text{and } \sum_{i=1}^{q+1} a_i^{-1} B_i = Q A^{-1} Q'$$

$$\text{where } A = \text{diag}(a_1 I_{p_1}, a_2 I_{p_2}, \dots, a_{q+1} I_{q+1})$$

So

$$\begin{aligned}\hat{\beta} &= (X' \Sigma^{-1} X)^{-1} X' \Sigma^{-1} Y \\ &= (X' [\sum a_i^{-1} (B_i + C_i)] X)^{-1} X' [\sum a_i (B_i + C_i)] Y \\ &= (X' [\sum a_i^{-1} B_i] X)^{-1} X' [\sum a_i^{-1} B_i] Y\end{aligned}$$

$$= (X' Q A^{-1} Q' X)^{-1} X' Q A^{-1} Q' \gamma$$

$$= (Q' X)^{-1} A (X' Q)^{-1} X' Q A^{-1} Q' \gamma$$

$$= (Q' X)^{-1} Q' \gamma$$

Note  $X = \sum_{i=1}^{q+1} B_i X = Q Q' X$

So  $Q = X (Q' X)^{-1}$  and  $Q' = (X' Q)^{-1} X'$

$$= [(X' Q)^{-1} X' X]^{-1} Q' \gamma$$

$$= [X' X]^{-1} X' Q Q' \gamma$$

$$= (X' X)^{-1} X' \gamma \quad b/c \quad Q Q' X = X$$


---

The log-likelihood function is

$$\ell = -\frac{1}{2} \log \left| \sum_{i=1}^{q+1} a_i (B_i + C_i) \right|$$

$$-\frac{1}{2} (y - X\beta)' \sum a_i^{-1} (B_i + C_i) (y - X\beta)$$

Note that

$$\left| \sum a_i (B_i + C_i) \right| = \prod_{i=1}^{q+1} a_i^{p_i + r_i}$$

b/c  $\sum (B_i + C_i) = I = P P'$   $\neq P$  is orthogonal

$$\text{so } \sum a_i (B_i + C_i) = P A^* P'$$

where  $A^* = \text{blockdiag} \left[ a_i I_{r_i + r_i} \right]$

It follows that

$$\begin{aligned} \frac{\partial \ell}{\partial a_i} &= -\frac{1}{2}(r_i + r_i) \frac{1}{a_i} \\ &\quad + \frac{1}{2a_i^2} (y - x\hat{\beta})'(B_i + C_i)(y - x\hat{\beta}) \end{aligned}$$

Substituting in  $\hat{\beta}$  and solve for  $a_i$

$$\begin{aligned} a_i &= \frac{1}{r_i + r_i} (y - x\hat{\beta})(B_i + C_i)(x - x\hat{\beta}) \\ &= y' [I - x(x'x)^{-1}x] (B_i + C_i) [x - x\hat{\beta}] y \\ &= \frac{y' C_i y}{r_i + r_i} \end{aligned}$$

b/c  $C_i x = 0$  and  $x'(x'x)^{-1}x' B_i = B_i$ .

The latter follows from  $Q = x(Q'x)^{-1}$

$$\text{so } x(x'x)^{-1}x' Q = x(Q'x)^{-1} = Q$$

$$\text{so } x(x'x)^{-1}x' Q_i = Q_i$$

$$\text{so } x(x'x)^{-1}x' Q_i Q_i' = Q_i Q_i'$$

$$\text{implies } x(x'x)^{-1}x' B_i = B_i \text{ b/c } B_i = Q_i Q_i'$$

$$\begin{aligned}
 E[Y' C_i Y] &= \text{tr}(C_i \Sigma) + \beta' \chi' C_i \chi \beta \\
 &= \text{tr}\left(C_i \sum_{j=1}^{q+1} \alpha_j (B_j + C_j)\right) + 0 \\
 &= \text{tr}(\alpha_i C_i) = \alpha_i r_i
 \end{aligned}$$

EMS estimate of  $\alpha_i$  is

$$\hat{\alpha}_i = \frac{Y' C_i Y}{r_i}$$

EMS estimate is unbiased

---

### RCB Design

$$\alpha_1 = r + \sigma_B^2 + r \sigma_{BT}^2 + \sigma_\epsilon^2$$

$$\alpha_2 = r \sigma_{BT}^2 + \sigma_\epsilon^2$$

$$\alpha_3 = \sigma_\epsilon^2$$

ML estimates

$$\hat{\alpha}_1 = \frac{Y' C_1 Y}{1 + (b-1)} \quad C_1 = C_b \otimes \bar{J}_x \otimes \bar{J}_r$$

$$\hat{\alpha}_1 = \frac{tr}{b} \sum_r^b (\bar{Y}_{i..} - \bar{Y}_{...})^2$$

$$\hat{\alpha}_2 = \frac{Y' C_2 Y}{(t-1) + (b-1)(t-1)} \quad C_2 = C_b \otimes C_t \otimes \bar{J}_r$$

$$\hat{\alpha}_2 = \frac{r}{b(t-1)} \sum_{i,j} (\bar{Y}_{ij..} - \bar{Y}_{i..} - \bar{Y}_{.j..} + \bar{Y}_{...})^2$$

$$\begin{aligned}
 \hat{\alpha}_3 &= \frac{\gamma' C_3 \gamma}{6t(r-1)} \quad C_3 = I_b \otimes I_c \otimes C_r \\
 &= \frac{1}{6t(r-1)} \sum_i \sum_j \sum_k (\gamma_{ijk} - \bar{\gamma}_{ijk})^2
 \end{aligned}$$