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- $\tilde{Y} \sim N_n(\mu, \Sigma)$ has a multivariate normal distr. if and only if $\underline{\alpha}' \tilde{Y}$ is univariate normal for all constant vectors $\underline{\alpha}$.
- Linear transformations of normals are normal

$$\tilde{Y} \sim N_n(\mu, \Sigma)$$

$$\tilde{X} = \underbrace{B\tilde{Y} + \underline{b}}_{m \times n} \sim N_m(B\mu + \underline{b}, B\Sigma B')$$

- e.g. $Z \sim N(0, 1)$, $\tilde{Y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} Z + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

then $\tilde{Y} \sim N\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\right]$

$$\sim N\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\right] \Sigma_Y$$

$$\Sigma_Y \text{ is p.s.d.}, \Sigma_Y = PDP'$$

$$P = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

Values of \tilde{Y} lie on a line (hyperplane)

$$\begin{aligned} P_2'(\tilde{Y} - \mu) = 0 &\Leftrightarrow 2y_1 - y_2 + 1 = 0 \\ &\Leftrightarrow y_2 = 2y_1 + 1 \end{aligned}$$

- $\underline{Y} \sim N_n(\underline{\mu}, \Sigma)$ then

$$m_{\underline{Y}}(\underline{t}) = \exp \left\{ \underline{t}' \underline{\mu} + \frac{1}{2} \underline{t}' \Sigma \underline{t} \right\}$$

- If $\Sigma > 0$ (Σ is p.d.) then

$$f_{\underline{Y}}(\underline{y}) = (2\pi)^{-n/2} \exp \left\{ -\frac{1}{2} (\underline{y} - \underline{\mu})' \Sigma^{-1} (\underline{y} - \underline{\mu}) \right\}$$

$\underline{y} \in \mathbb{R}^n$

If $\Sigma \geq 0$ (Σ is p.s.d.), $\text{rank}(\Sigma) = r < n$

$$\begin{aligned} f_{\underline{Y}}(\underline{y}) &= (2\pi)^{-r/2} \left(\prod_{i=1}^r \lambda_i \right)^{-1/2} \\ &\quad \times \exp \left\{ -\frac{1}{2} (\underline{y} - \underline{\mu})' P_r D_r^{-1} P_r' (\underline{y} - \underline{\mu}) \right\} \\ &\quad \underline{y} \in \{ \underline{y} : P_r' (\underline{y} - \underline{\mu}) = 0 \} \end{aligned}$$

where $P = (P_r, P_{\Sigma})$, $D_r = \text{diag}(\lambda_i)_{i=1}^r$

$$\Sigma = P D P'$$

If $\Sigma^- = P_r D_r^{-1} P_r'$ then (check)

$$\Sigma^- \Sigma \Sigma^- = \Sigma^- \quad \text{and} \quad \Sigma \Sigma^- \Sigma = \Sigma$$

Σ^- is a generalized inverse of Σ

- If $\underline{Y} = (\underline{y}_1', \underline{y}_2')'$ is multivariate normal

$$\underline{Y} \sim N_n(\underline{\mu}, \Sigma) = N_n\left[\begin{pmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right]$$

then \underline{y}_1 and \underline{y}_2 are independent iff $\Sigma_{12} = 0$

- $A\underline{Y} + \underline{a}$ and $B\underline{Y} + \underline{b}$

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \underline{Y} + \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} = \begin{pmatrix} A\underline{Y} + \underline{a} \\ B\underline{Y} + \underline{b} \end{pmatrix}$$

$$\text{cov}\left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \underline{Y}\right] = \begin{bmatrix} A\Sigma A' & A\Sigma B' \\ B\Sigma A' & B\Sigma B' \end{bmatrix}$$

Independent "if" $A\Sigma B' = 0$

Conditional Distributions

$$\underline{Y} \sim N_n(\underline{\mu}, \Sigma) \quad \underline{\mu} = (\underline{\mu}_1', \underline{\mu}_2')'$$

$$\underline{Y} = (\underline{y}_1', \underline{y}_2')' \quad \underline{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

What is the conditional distribution of \underline{y}_2 given \underline{y}_1 ?

Assume (for now) that $\Sigma_{11} > 0$

Lemma: Define $\gamma_{2.1} = \underline{\gamma_2 - \Sigma_{21}\Sigma_{11}^{-1}\gamma_1}$

Then $\gamma_{2.1} \sim N(\mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1, \Sigma_{22.1})$

where $\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$

independently of γ_1 .

$$\text{Proof } \begin{pmatrix} \gamma_1 \\ \gamma_{2.1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -\Sigma_{12}\Sigma_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

$$\begin{aligned} \text{cov} \begin{pmatrix} \gamma_1 \\ \gamma_{2.1} \end{pmatrix} &= \begin{pmatrix} I & 0 \\ -\Sigma_{12}\Sigma_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} I & -\Sigma_{11}^{-1}\Sigma_{12} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \underline{\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}} \end{pmatrix} \end{aligned}$$

Theorem

$$\gamma_2 | \gamma_1 = y_1 \sim N \left[\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(y_1 - \mu_1), \Sigma_{22.1} \right]$$

Proof: $\gamma_{2.1}$ is independent of γ_1 . Hence its conditional distr given $\gamma_1 = y_1$ is the same as its marginal distr.

But $\gamma_2 = \gamma_{2.1} + \Sigma_{21}\Sigma_{11}^{-1}\gamma_1$ given $\gamma_1 = y_1$ is normal with mean

$$\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}\mu_1 + \Sigma_{21}\Sigma_{11}^{-1}y_1 = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(y_1 - \mu_1)$$

and covariance $\Sigma_{22..}$

Example: $Y_1, \dots, Y_n \sim \text{iid } N(\alpha, \sigma^2)$

Equivalently $Y \sim N_n \left[\alpha \mathbf{1}_n, \sigma^2 I_n \right]$

Consider the transformation $\begin{bmatrix} A \\ B \end{bmatrix} Y$

where $A = \frac{1}{n} \mathbf{1}'_n$ and $B = I_n$

$$\begin{bmatrix} A \\ B \end{bmatrix} Y = \begin{bmatrix} \frac{1}{n} \mathbf{1}'_n \\ I_n \end{bmatrix} Y \sim N \left[\begin{pmatrix} \alpha \\ \alpha \mathbf{1}_n \end{pmatrix}, \begin{bmatrix} \sigma^2/n & \sigma^2 \mathbf{1}'_n \\ \sigma^2 \mathbf{1}_n & \sigma^2 I_n \end{bmatrix} \right]$$

What is the conditional distn. of $Y | \bar{Y} = \bar{y}$?

$$Y | \bar{Y} = \bar{y} \sim N \left[\alpha \mathbf{1}_n + \frac{\sigma^2}{n} \mathbf{1}_n \left(\frac{\sigma^2}{n} \right)^{-1} (\bar{y} - \alpha), \sigma^2 I_n - \frac{\sigma^2}{n} \mathbf{1}_n \left(\frac{\sigma^2}{n} \right)^{-1} \frac{\sigma^2}{n} \mathbf{1}'_n \right]$$

$$\sim N \left[\bar{y} \mathbf{1}_n, \underbrace{\sigma^2 (I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}'_n)}_{I_n - \bar{J}_n = C_n} \right]$$

Example: $Y \sim N_n (\alpha \mathbf{1}_n, \sigma^2 I_n)$

$$\text{Recall } Y' C_n Y = \sum_{i=1}^n (Y_i - \bar{Y})^2$$

Also C_n is idempotent, $C_n^2 = C_n$

and $C_n \mathbf{1}_n = \mathbf{0}$

$$\begin{aligned} \text{cov}(C_n Y, \frac{1}{n} \mathbf{1}'_n Y) &= C_n (\sigma^2 I_n) \frac{1}{n} \mathbf{1}'_n \\ &= \sigma^2 C \mathbf{1} \frac{1}{n} = 0 \end{aligned}$$

Hence CY and \bar{Y} are independent!

Hence $Y'CY$ is indep. of \bar{Y} (why?)

$$Y'CY = Y'C'C'CY = (CY)'(CY)$$

- Previously $C_n = P_n P_n'$ where P_n $n \times (n-1)$ consists of cols. $2, \dots, n$ of the Helmert matrix

$$Y'CY = (Y'P_n)(P_n'Y) = X'X$$

$$\text{where } X = P_n'Y \sim N(P_n'(\alpha \mathbb{1}_n), \sigma^2 P_n' I_n P_n) \\ \sim N(0, \sigma^2 I_{n-1})$$

$$\text{So } X'X \sim \sigma^2 \sum_{i=1}^{n-1} Z_i^2 \quad \text{where } Z_i \sim \text{iid } N(0, 1)$$

$$\text{i.e. } \sum_{i=1}^{n-1} (Y_i - \bar{Y})^2 \sim \sigma^2 \chi_{n-1}^2$$

Distributions of Quadratic Forms

- Let X_1, \dots, X_n be independent $N(\mu_i, 1)$ $i=1, \dots, n$. Define $Y = \sum_{i=1}^n X_i^2$ and $S = \sum_{i=1}^n \mu_i^2$.

Then, we say Y has a non-central chi-squared distn with n degrees of freedom and non-centrality S .

- Write $Y \sim \chi_n^2(S)$

Write $Y \sim c \chi_n^2(S)$ if $Y/c \sim \chi_n^2(S)$

Theorem If $\mathbf{Y} \sim N_n(\mathbf{0}, \mathbf{I}_n)$. Then

$$\mathbf{y}' \mathbf{A} \mathbf{y} \sim \chi_p^2(0) \quad (\text{central chisquare})$$

if and only if \mathbf{A} is idempotent with rank r

Proof (if) If $\mathbf{A}^2 = \mathbf{A}$ and $\text{rank}(\mathbf{A}) = r$. Then

\mathbf{A} has eigenvalue 1 with multiplicity r and 0 with multiplicity $n-r$ So by SVD

$$\mathbf{A} = \mathbf{P} \mathbf{D} \mathbf{P}' = \mathbf{P}_r \mathbf{P}_r'$$

where \mathbf{P}_r $n \times r$ has cols. corresponding to the non-zero eigenvalues. Then

$$\mathbf{x} = \mathbf{P}_r' \mathbf{Y} \sim N_r(0, \mathbf{P}_r' \mathbf{I}_n \mathbf{P}_r) = N_r(0, \mathbf{I}_r).$$

So

$$\mathbf{y}' \mathbf{A} \mathbf{y} = \mathbf{y}' \mathbf{P}_r \mathbf{P}_r' \mathbf{y} = \mathbf{x}' \mathbf{x} = \sum_{i=1}^r x_i^2 \sim \chi_r^2(0)$$

because x_1, \dots, x_r are iid $N(0, 1)$