

Linear Mixed Model

$$Y = X\beta + Zu + E$$

↑ ↑ ↑ ↑
response fixed random error
 effects effects

Matrix notation and results

$\underline{A}, \underline{T}, \underline{X}, \underline{Y}, \underline{x}, \underline{y}, \underline{u}$ matrices, vectors
(bold in the book)

γ, u, E scalar random variables

r, s, c, n scalar constants

• $r \times s$ matrix $A = (a_{ij})$

a_{ij} = element in i th row, j th col.

• transpose $A' = (a_{ji})$ $s \times r$

• identity, $\underline{I}_{n \times n}$

• one-vector $\underline{1}_n$ $n \times 1$ vector of 1s

• one matrix $\underline{J}_{n \times s}$, $\underline{J}_{r \times s} = \underline{1}_r \underline{1}'_s$

$$\underline{\overline{J}}_n = \frac{1}{n} \underline{J}_{n \times n} = \frac{1}{n} \underline{1}_n \underline{1}'_n$$

• matrix operations: addition, subtraction
multiplication

• Square matrices: inverse?, non-singular
diagonal, determinant, symmetric

• Linear dependence and rank

$$\underline{A}_{n \times s} = [\underline{a}_1, \dots, \underline{a}_s]$$

The cols. are linear dependent if there
exists $\underline{x} = [x_1, \dots, x_s]' \neq 0$ such that

$$\underline{A} \underline{x} = x_1 \underline{a}_1 + \dots + x_s \underline{a}_s = 0$$

Otherwise the cols are linearly independent

The rank of A is the number of L/N cols.

• Quadratic Form: n -variate functions

$$f(x_1, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j = \underline{x}' \underline{A} \underline{x}$$

is called quadratic form in \underline{x}

Without loss of generality (WLOG)
we can assume A is symmetric (Why?)

eg.

$$\underline{x}' \underline{I} \underline{x} = \underline{x}' \underline{x} = \sum_{i=1}^n x_i^2$$

$$\underline{x}' \underline{J} \underline{x} = \underline{x}' \mathbf{1} \mathbf{1}' \underline{x} = (\sum x_i)^2$$

$$\bullet C = \underline{I} - \underline{J}$$

$$\begin{aligned} \underline{x}' C \underline{x} &= \sum x_i^2 - \frac{1}{n} (\sum x_i)^2 \\ &= \sum (x_i - \bar{x})^2 \end{aligned}$$

C is called the centering matrix

$$C \underline{x} = (I - \bar{J}) \underline{x} = \underline{x} - \frac{1}{n} \mathbf{1} \mathbf{1}' \underline{x}$$

$$= \underline{x} - \frac{1}{n} \bar{x}$$

- Orthogonal Matrix: P $n \times n$
orthogonal if $P'P = PP' = I$

write $P = [P_1, \dots, P_n]$ then

$$P_i' P_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

columns of P are orthonormal

- \underline{x} and \underline{y} are orthogonal if $\underline{x}'\underline{y} = 0$
If \underline{x} and \underline{y} are orthogonal they are LIN

- Helment matrix

$$\underline{a}'_1 = \underline{1}'$$

$$\underline{a}'_2 = (1, -1, 0, \dots, 0)$$

$$\underline{a}_3 = (1, 1, -2, 0 \dots 0)$$

:

$$\underline{a}_n = (1, 1, \dots, 1, -(n-1))$$

Clearly $\underline{a}_i \cdot \underline{a}_j = 0$ if $i \neq j$

Define $R_i = \frac{\underline{a}_i}{\sqrt{\underline{a}_i \cdot \underline{a}_i}}$, then $R_i \cdot R_i = 1$

$P = [R_1, \dots, R_n]$ is the Helmholtz matrix

Determinant of Square matrices

$A_{n \times n}$

$$|A| = \sum_{i=1}^n a_{ij} C_{ij}, \quad C_{ij} = (-1)^{i+j} |M_{ij}|$$

M_{ij} is the minor of a_{ij} formed by eliminating row i and col. j

- $|A'| = |A|$
- $|A^{-1}| = 1/|A|$ if $|A| \neq 0$
- $|AB| = |A||B|$
- $|aB| = a^n |B|$
- $A\underline{x} = 0$ has a non-zero solution if and only if A is singular or equivalently $|A| = 0$

Eigenvalues and vectors: $A_{n \times n}$

λ is an eigenvalue of A if

$$A\underline{x} = \lambda\underline{x} \quad (*)$$

for some non-zero \underline{x} . Then \underline{x} is called an eigenvector.

(*) is equivalent to

$$(A - \lambda I)\underline{x} = \underline{0}$$

and hence to $|A - \lambda I| = 0$ / characteristic equation

$|A - \lambda I|$ is a polynomial of degree n in λ . It has n roots

A has n eigenvalues $\lambda_1, \dots, \lambda_n$ and n associated eigenvectors x_1, \dots, x_n

Fact: Real symmetric matrices only have real eigenvalues.

Spectral Decomposition (of a symmetric matrix)

If $A_{n \times n}$ is real symmetric then

$$A = P D P'$$

where P is orthogonal with cols. equal to eigenvectors, and D is diagonal with the corresponding eigenvalues

Equivalently
$$A = \sum_{i=1}^n \lambda_i p_i p_i'$$

If A is real and symmetric

$$\begin{aligned}\operatorname{tr}(A) &= \operatorname{tr}(PDP') = \operatorname{tr}(DP'P) = \operatorname{tr}(D) \\ &= \sum_{i=1}^n \lambda_i\end{aligned}$$

$\operatorname{tr}(AB) = \operatorname{tr}(BA)$ — trace is invariant to cyclical permutations

$$\begin{aligned}|A| &= |PDP'| = |P||D||P'| \\ &= |D||PP'| = \prod_{i=1}^n \lambda_i\end{aligned}$$

$$1 = |PP'| = |P||P'| = |P|^2 \Rightarrow |P| = 1 \text{ or } -1$$

←

eg. Consider $G = aI_n + bJ_n$

$$\begin{aligned}G \underline{1} &= a \underline{1}_n + b J \underline{1}_n \\ &= a \underline{1} + b \underline{1} \underline{1}' \underline{1} = a \underline{1} + nb \underline{1} \\ &= (a + nb) \underline{1}\end{aligned}$$

$\underline{1}_n$ is an eigenvector of G with eigenvalue $a + nb$

Consider p_2, \dots, p_n from the Helmholtz matrix

$$\begin{aligned} G p_i &= (a I + b J) p_i = a p_i + b \frac{1}{n} \mathbf{1} \mathbf{1}' p_i \\ &= a p_i \end{aligned}$$

p_2, \dots, p_n are eigenvectors of G ,
with eigenvalue a (with multiplicity
 $n-1$)

Consider $a=1$, $b=\frac{1}{s}$

$$G = I - \frac{1}{s} J = C$$

$$a - sb = 0, \quad a=1$$

$$C^2 = (I - \frac{1}{s} J)^2 = \dots = C$$